Compliance of Rigid Arbitrary Shape Foundations Using 1 DOF BEM

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Abstract

In this study the relationship between the dynamic force and displacement (impedance or compliance) is evaluated for rigid foundations with arbitrary shape resting on a half-space medium, consisting of homogeneous, isotropic, linear elastic materials with constant Boundary Element Method, (1 DOF). Green's function is computed for half-space and presented in explicit form. By using BEM formulation the stress beneath foundations and compliance of them are obtained. The vertical compliance of a rigid circular disc which is calculated by this method is compared by semi-closed form solution and verifies the accuracy and efficiency of this method. The vertical and rocking compliance functions for rigid rectangular and some arbitrary shape foundations are obtained. And also, the variation of stress distribution pattern beneath square and circular foundations with frequency is studied and these results are compared with the results of the other methods.

KEY WORD: Green’s function – stress distribution – 1 DOF BEM – Lamb’s solution - Boussinesq’s solution

1. INTRODUCTION

An important step in the study of the dynamic interaction between structures and the supporting medium is the evaluation of foundation compliance functions. These functions play a key role in the design of foundations for oscillatory machines and in computations of structural response to earthquake excitations. Some studies on the dynamic response of foundations have been restricted to analysis of rigid circular footings Veletsos and Y.T.Wei [19], Luco and Westmann [14] or to the two-dimensional rigid strip foundation Karasudhi et,al [3], Luco and Westmann [15]. In these cases the resulting mixed boundary value problems may be reduced by standard techniques to solution of Freehold integral equations. For other geometrical shapes of the foundations, the problem was approximately solved by defining an equivalent circular base, or by assuming a certain stress distribution in the contact plane between the foundation and the soil. Kobori and others was obtained the dynamic compliance functions for rectangular foundations Thomson and Kobori [18], Kobori and Suzuki [10].

In soil surface interaction of due to complexity of geometry and boundary conditions of this problem, the numerical approaches are used to analyze an arbitrary shaped rigid foundation. Most of the numerical methods are based on dividing the contact plane between the foundation and the ground surface into finite number of elements in which the stress distributed are assumed to be known but stress values are unknown. In these methods by using above assumptions and boundary condition and also equilibrium equation and compatibility of deformations, obtained a system of linear simultaneous equations. Solving this equations system obtained values of stress.

Wong and Luco [21] use above-mentioned approach, to evaluate compliance of arbitrary shape foundations. Their methods had two difficulties, First: influence function (equivalence with Green’s function) used by them involve numerical double integration; whose one of its upper limits is infinite and a huge computation cost time need due to evaluating \( n^2 \) (number of elements) influence function. Second: it is impossible to develop for other shapes of elements and higher orders elements; due to they doesn’t using explicitly BEM formulation. Also they used a particular solution obtained by Thomson and Kobori [10], where evaluate the, occurring displacement at an arbitrary point, when a uniform harmonic stress acts on the surface of the rectangular element.
Savidis and Richter [17] also analyzed interaction problems of two buildings having rigid, rectangular foundations, by use of a similar method to Wong and Luco approach. Kitamura and

Sakurai [7] use similar method, they divide contact plane into some rectangular elements, that contact stress assumed to be uniformly distributed over each element, and evaluate displacement of nodes where is in the center of elements.

and Abascal [1], Karabalis and Beskos [6] use BEM formulation to obtained compliance of arbitrary rigid foundation in time and frequency domain. They use infinite-space Green's function for frequency domain, and need greet number of elements to modeling of free surface, and can not modeling whole of free surface, hence results is not very accurate.

In this study, the Green's function is computed for half-space elastic medium and presented in explicit form. Then by using BEM formulation, for rectangular constant elements the stress in contact plane is obtained, and also it can be computed the compliance of rigid arbitrary shape foundation resting on the liner, homogenous, isotropic, half-space elastic medium, with any fraction on the contact plane. In this research Compliance of circular, square, and some other shapes foundations were evaluated, and compared with the other studies.

The Present method is better than Wong and Luco [21] method, due to using explicit form of Green's function with high accuracy and reduces the computation cost time and Also it can be developed easily to other shapes of element and high orders due to using of explicitly BEM formulation. as compare with Kitamura and Sakurai, this method is applicable in wide range of frequency, and better accuracy than their method due to assumption stress distribute uniformly over the elements.

Present method needs fewer elements than and Abascal, Beskos and Karabalis because of using half-space Green's function, and must be model by only contact plane of foundation and ground.

2. BASIC EQUATIONS

Lamb [11] has derived the vertical displacement at an arbitrary point on the surface of a half-space isotropic elastic medium, due to a concentrated periodical force, \( Q e^{i\omega t} \) being applied vertically at the origin, (Green’s function) as follows,

\[
G_z(\alpha, \nu) = u_z = -\frac{Q e^{i\omega t}}{2\pi \mu} \left( \frac{\alpha}{r} \right)^2 \frac{1}{F(\zeta)} \int_0^\infty J_0(\alpha \zeta) d\zeta
\]

where \( F(\zeta) = (2\zeta^2 - 1)^2 \sqrt{\zeta^2 - \gamma^2} \), \( \gamma = \sqrt{(1-2\nu)/(1-\nu)} \); \( \alpha = \omega r / C_s \), \( r = \) distance; \( Q = \) amplitude of periodical force; \( \omega = \) circular frequency of harmonic load; \( \mu = \) shear modulus; \( C_s = \) propagation velocity of shear wave; \( \nu = \) Poisson ratio; \( t = \) time and \( J_0(\alpha \zeta) = \) zero order of Bessel's function of the first kind.

Eq. 1 can be rewritten as:

\[
G_z(\alpha, \nu) = -\frac{1 - \nu}{2\pi \mu} \frac{Q e^{i\omega t}}{r} CF(\alpha, \nu)
\]

where \( CF(\alpha, \nu) \) a complex function from \( \alpha \), and \( \nu \) that defined as

\[
CF(\alpha, \nu) = \frac{\alpha}{1 - \nu} \int_0^\infty \frac{\zeta^2 - \gamma^2}{F(\zeta)} J_0(\alpha \zeta) d\zeta
\]

Eq. 2 in other word, mention relationship between dynamic and static (Boussinesq's solution) Green's functions with a coefficient function as \( CF(\alpha, \nu) \), we obtained, \( CF(\alpha, \nu) \) tend one when \( \alpha \) tends to zero.

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3. **NUMERICAL EVALUATION OF SEMI-INFINITE SPACE GREEN’S FUNCTION**

We examine now some of the problems encountered in the evaluation of Eq. 3. The function $F(\zeta)$ leads to two branch-point singularities and a Rayleigh pole, all lying along the positive $\zeta$-axis with their image reflection along the negative $\zeta$-axis as shown in Fig. 1. Ewing et al. [2] shows that the branch barriers are hyperbolases in second and fourth quadrants as shown by dashed curve in Fig. 2. Thus for any value of $\alpha$ the $\zeta$ integration of Eq. 3 may be expressed in terms of the contour integration around the single-valued path shown in Fig. 2, which encloses the Rayleigh pole on the positive $\zeta$-axis. The analytic evaluation of this integral is, however, extremely difficult, and hence a procedure for it's numerical evaluation by digital computation was developed. $F(\zeta)$ and integrand of Eq. 3 has the characteristics given in Table 1.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$F(\zeta)$</th>
<th>Integrand</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq \zeta \leq 1/\sqrt{4}$</td>
<td>Real</td>
<td>Imaginary</td>
</tr>
<tr>
<td>$1/\sqrt{4} \leq \zeta \leq 1$</td>
<td>Complex</td>
<td>Complex</td>
</tr>
<tr>
<td>$1 \leq \zeta &lt; 1.072356$</td>
<td>Positive real</td>
<td>Real</td>
</tr>
<tr>
<td>Pole $\zeta_0 = 1.072356$</td>
<td>Zero</td>
<td>Real</td>
</tr>
<tr>
<td>$1.072356 &lt; \zeta \leq \infty$</td>
<td>Negative real</td>
<td>Real</td>
</tr>
</tbody>
</table>

**Fig. 1 Integration contour in complex plane.**

Separate integration’s were performed for each of the regions in Table 1 and a special technique for integrating over the pole, described by Longman [12] and Quinlan [16] of splitting the integrand into an odd and an even function, also for computing infinite integrals of oscillatory function replacing slowly convergence series was used.

The results that are obtained by the digital computer are equal to the Cauchy’s principal value of the integral of Eq. 3. To avoid standing waves, or to insure that only out going waves are present, it will be found necessary to subtract one half of the residue at the Rayleigh pole, Ewing [2] and Wei [20]. Thus $-i\pi R$ must be added to equation (3) as follow:

$$-i\pi R = -i\pi \frac{\zeta_0 \sqrt{\gamma^2 - \zeta^2}}{F'(\zeta_0)} J_0(\alpha\zeta_0)$$

where

$$F'(\zeta_0) = \frac{d}{d\zeta} F(\zeta) \bigg|_{\zeta = \zeta_0}$$

where $\zeta_0 = \text{point that } F(\zeta)$.

By using the above method for any value of $\alpha$, $CF(\alpha, \nu)$ can be obtained. $CF(\alpha, \nu)$ is expressed in following form:
\[ CF(\alpha, \nu) = f_1 + if_2 \]

\[ CF(\alpha, \nu) = \sqrt{f_1^2 + f_1^2} e^{-i\varphi} \]

where \( \varphi = \tan^{-1}(f_2 / f_1) \).

4. ANALYSIS PROCEDURE

Consider an arbitrary shape of a structure resting on the surface of a semi-infinite medium. For rigid surface foundation by using the half-space Green function and assuming that the vertical and rocking modes are uncoupled, Wong and Luco [21], and also by using Fig. 5, we can drive vertical displacement of an arbitrary point of half-space surface as,

\[ u^b_z (\alpha) = \int_{\Gamma} \frac{G_z(\alpha)}{\sigma^b_z(\alpha^*)} d\Gamma \]

where \( \alpha \) and \( \beta \) are arbitrary point of half-space surface and contact plane of foundation and earth respectively; \( r^* \) = distance between point \( \alpha \) and \( \beta \); \( \alpha^* = r^* \omega / C_s \); \( \sigma^b_z \) = stress in the contact plane of foundation and earth; and \( \Gamma \) = contact plane of foundation and earth. By solving integral Eq. 7 with boundary condition which is written as fallow,

\[ \begin{cases} u^a_z(\alpha^*) = -e^{i\omega \tau} & a \in \Gamma \\ \sigma^b_z(\alpha^*) = 0 & a \notin \Gamma \end{cases} \]

\( \sigma^b_z \) and compliance are obtained. Luco and Westman [15] used this method for circular foundation. This method may by use simple shape and almost impossible use for arbitrary shape foundations.

A numerical method is developed for solving this problem. As the Fig. 6 shows the contact area between the foundation and the ground dividing in to the number of elements and assume stress distribution uniform within each element by unknown value, and then the Eq. 7 for center of elements(nodes of elements) is rewritten as fallow:

\[ u^p_z = \sum_{q=1}^{N} \sigma^b_z \left( \int_{A_q} G_z(\alpha^*) dA_q \right) \]

where \( u^p_z = \) vertical displacement of \( p \) th node; \( \sigma^b_z = \) vertical stress in the \( q \) th element; \( A_q = \) area
of the element; \( G_z(\alpha^*) = \) Green’s function; \( \alpha^* = r^* \omega / C_s \) and \( r^* = \) distance between node p and differential element.

In Eq. 9, the term which in parenthesis is rewritten as Eq. 10.

\[
C_{pq} = \int G_z(\alpha^*) dA_q = -\frac{1 - \nu}{2\pi\mu} Q e^{i\alpha} \int \frac{CF(\alpha^*, \nu)}{r^*} dA_q
\]  

Then by replacing \( C_{pq} \) in Eq. 9, it is simplified to Eq. 11:

\[
u_p^q = C_{pq} \sigma^q_z \quad ; \quad p, q = 1, 2, \ldots, n
\]

By calculating Eq. 9 for n nodes, \( C_{pq} \) in Eq. 11 is obtained. When the foundation is assumed to be rigid and satisfying the compatibility equations at the contact plane, the following relationships for vertical and rocking vibration, respectively, are as follow:

for vertical vibration

\[
u_p^v = \Delta_z e^{i\alpha}
\]

(12)

for rocking vibration

\[
u_p^r = \theta \cdot d_p e^{i\alpha}
\]

(13)

Where \( \Delta_z = \) amplitude of vertical displacement, \( \theta = \) rotation angle around line D, \( d_p = \) distance from line D.

Solving Eq. 11 under the conditions of Eq. 12 or 13 gives the contact pressure distributions under rigid foundations, and the results are described as the complex amplitude. By using equilibrium equation, the vertical and rocking impedance are obtained as follow:

\[
K_{VV} = \frac{P_z}{\Delta_z} \quad \text{for vertical vibration}
\]

(14)

\[
K_{MM} = \frac{M_D}{\theta} \quad \text{for vertical vibration}
\]

(15)

Numerical Evaluation of Integrals in Analysis Procedure

Eq. 10 can rewrite in the following form for qth rectangular element with Length 2bq and width 2cq, as shown in Fig. 7.i

\[
C_{pq} = \frac{1 - \nu}{2\pi\mu} e^{i\alpha} \int \left( \int \frac{\sqrt{x_q^2 + b_q^2}}{x_q^2 - c_q^2} \frac{CF(\alpha^*, \nu)}{r^*} dy \right) dx
\]  

(16)
where \( x_q \) and \( y_q \) are coordinate of node \( Q \) (center of element); \( r = \) distance between differential element and node \( p \); \( b_q \) and \( c_q \) are half of length and wide of element \( q \) and \( \alpha = \Omega / \Omega_q \). 
If node \( p \) is not equals with node \( q \), then Eq. 16 is computed with a numerical method e.g. Gauss-Lagrange, but if \( p \) is equal with \( q \), the evaluation of Eq. 16 is not very easy, because of it will be singular. For explanation this, we define a polar coordinate system in the center of element which is shown in Fig. 7.ii and rewrite Eq. 16 as follow:

\[
C_{pq} = \frac{1 - \nu}{\pi \mu} e^{i \alpha} \left( \frac{\beta_1}{\beta_1} \int_{\beta_1}^{\beta_2} \int_{\beta_2}^{\beta_1} CF(\alpha, \nu) drd\theta + \int_{\beta_1}^{\beta_2} \int_{\beta_2}^{\beta_1} CF(\alpha, \nu) drd\theta \right)
\]

(17)

where \( \beta_1 = \tan^{-1} \left( \frac{c_q}{b_q} \right) \) and \( \beta_2 = \tan^{-1} \left( \frac{b_q}{c_q} \right) \). As it is shown in Eq. 17, it is not singular equation and it can be evaluate by a numerical method.

5. RESULT AND COMPARISONS
To illustrate the method which is used above, the compliance of rigid foundation with different geometrical configurations has been evaluated. The shear stress on the contact plane, for vertical and rocking vibrations have been assumed to be zero Wong and Luco, [21].

For a first example, the vertical compliance, \( C_{VV}(\alpha_0) \) for a rigid circular foundation were obtained by considering 52 equal square elements which is shown in Fig. 8. The real and imaginary part's of compliance function is defined as fallow:

\[
\zeta_p = \mu \alpha_0 \Delta z
\]

(18)

where \( \mu = \) is the shear modulus for the elastic half-space; and \( a \) is the radius of the circular foundation, for different value of the dimension-less frequency \( \alpha_0 = a \omega / C_z \), The values of compliance are shown in Tab. 2; and Fig. 9. The corresponding values which are obtained by Luco and Westmann [14], Wong and Luco [21] by using different approach are also listed in Table 2. The agreement between results of present study and Luco and Westmann capital research is excellent.

The variation of stress distribution under rigid circular foundation with frequency was evaluated. Fig. 10 and Fig. 11 evaluate variation of absolute value and phase angle of stress beneath foundation, in the center and perimeter of foundation respectively. For the second example, the vertical compliance function, \( C_{VV}(\alpha_0) \) for a rigid square foundation of side \( 2b \) was evaluated for different values of dimensionless frequency \( \alpha_0 = ab / C_z \) and for a Poisson ratio \( \nu=1/3 \) as fallow:

\[
\zeta_p = \mu \alpha_0 \Delta z
\]

(19)
The results which are obtained by replacing the contact plane by 1, 4, 16 and 64 equal square element are shown in Fig. 11. The results indicate that at least 16 elements are necessary to account for the rigidity of the foundation. Thomson and Kobori [18] have evaluated the vertical compliance for rectangular foundations by assuming a uniform stress distribution on the contact area and by evaluating the displacement at the center of the foundation. As the figure 11 shows, the assumption of Thomson and Kobori for 1 element (1x1), can not represent accurately the vertical compliance for a rigid foundation. The comparison between vertical compliance of a square foundation (8x8), and the other studies is shown in Fig. 12.

If a uniform stress distribution is assumed over total contact area, then a better estimate of the compliance functions is obtained by considering the average displacement of the foundation, which is shown in Table 3. In this table the static vertical compliance for a rectangular foundation of sides 2b and 2c, that is obtained by Wei [20], considering the average displacement of the foundation for a uniform stress distribution, are compared with the results obtained by the method which is used in this study.

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>Re $C_{vv}(a_0)$</th>
<th>Im $C_{vv}(a_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.1670 0.1690 0.1692 0.0000 0.0000 0.0000</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.1540 0.1570 0.1547 0.0495 0.0500 0.0493</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.1210 0.1240 0.1217 0.0846 0.0870 0.0849</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>0.0815 0.0838 0.0819 0.0977 0.1030 0.0989</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>0.0488 0.0504 0.0490 0.0922 0.1010 0.0947</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.0286 0.0296 0.0284 0.0796 0.0903 0.0824</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Comparison of the vertical compliance for a rigid circular foundations ($\nu = 1/3$).

1. Values obtained by Luco & Westmann.
2. Values obtained by Wong & Luco.
3. Present study.
6. CONCLUSIONS

Green’s function for half-space in frequency domain was evaluated with a numerical method. Also a numerical approximate of Green’s function for half-space is determined. The results show that Green’s function yields is an efficient (accurate and speed) numerical method for computing stress under of foundation and the compliance of foundation resting on half-space. It can be extended to over the case of a multi-layered and viscous-elastic supporting medium and also for flexible foundations.

Numerical results were obtained for the vertical and rocking compliance for circular, square, and rectangular foundations, and also were compared with another researches. The stress distribution beneath of foundation and it’s variation with frequency was evaluated, and obtained it’s variation with frequency. The vertical compliance for some rigid foundation with arbitrary shape was determined and it was found that, the using of square equivalent or rectangular, could not be an efficient method.

7. REFERENCES

[1] Abascal R., "Dynamic of foundations"(1987); Ch.2, Vol.4, Topics in Boundary Element Research, 27-75 Edit C.A.Brebbia,


