A new wavelet-based method for determination of mode shapes: Experimental Results

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Abstract
In this article a new method is proposed to determine the mode shapes of linear dynamic systems from the results of wavelet analysis. A previously proposed method based on a modified Morlet wavelet function with an adjusting parameter is used to identify the natural frequencies and damping ratios of system. The mode shapes are obtained from the time signal of responses and the extracted natural frequencies from wavelet transform of response signals. The method is applied to a steel real beam excited by an impact force. It is shown that the extracted mode shapes are not scaled. Therefore, the mass change method is used for scaling of the mode shapes.

Keywords: Mode shapes, Wavelet transform, Natural frequency, Free responses.

1. INTRODUCTION

Estimation of the modal parameters in terms of natural frequencies, damping coefficients and mode shapes from experimental data is a fundamental problem in structural dynamics. The modal parameter identification methods may be categorized in to Single Degree Of Freedom (SDOF) methods and Multi Degrees Of Freedom (MDOF) methods. Pick peaking method, circle fit method and line fit method are the classical methods for modal parameter identification [1]. The recent method of three point finite difference method [2] gives more accurate results compared to the traditional methods. Least square complex exponential method [3], poly-reference time domain method [4], Ibrahim time domain method [5], automated parameter identification and order reduction for discrete time series [6] are among the MDOF methods for modal parameter determination [7]. The basis of most of these methods is Fourier analysis which transforms the time data to the frequency data. However, Fourier analysis cannot determine the modal parameters accurately in the noisy environments. Some methods consist of pre-filtering of the input signals can improve the results. Moreover, close modes may hardly be identified using the techniques based on the Fourier analysis. Recently, wavelet analysis has attracted researchers in applied physics and engineering as well as the other branches of science due to its powerful capability in analyzing a signal [8]. In contrast to the Fourier transform which has a uniform resolution in frequency domain, the wavelet transform has the property of double resolution in both the time and frequency domain. By using this property, the wavelet transform can be adjusted to analyze the non-stationary signals. Also, the strongly coupled modes can be identified by tuning the wavelets. Moreover, the inherent ability of wavelet transform in filtering out the noise contaminating a signal is an important advantage for identifying the modal parameters. In previous years, some researches have been conducted for identification of modal parameter using wavelet transform [9-13]. The input signals to these wavelet techniques are mostly the ambient time records without the knowledge of input force and consequently these methods are comparable to output-only techniques in modal testing.

Three methods for estimating the damping ratios based on the Continuous Wavelet Transform (CWT) were proposed in [14]. A procedure of identification of natural frequencies and damping ratios of the system from its free decays using wavelet transform was proposed in [15]. A modified Morlet wavelet function with adjusting parameter was proposed in [16] to improve the accuracy of identification. A modal parameter identification procedure using continuous wavelet transform including the mode shape identification has been proposed in [17]. An identification method for natural frequencies and damping ratios based on a modulated Gaussian wavelet transform from impulse response function is
presented in [18]. A study of weak nonlinearity using modified wavelet transform was presented in [19].

The contribution of this paper consists essentially of introducing a new method for identification of mode shapes from the wavelet analysis of response data for the systems with proportional viscous damping excited by an impact force. A previously proposed Morlet wavelet with an adjusting parameter is used for identification of the natural frequencies and damping ratios. It is shown in [16] that the accuracy of modal parameter identification from wavelet can be improved by adjusting the proposed parameter. In this paper, a criterion based on Route Mean Square (RMS) method is used for comparison of the regenerated time signal and the original signal in order to tune the adjusting parameter in the wavelet.

The new method for mode shape identification is experimentally investigated on a real case study of a beam. As the method is based on the output only data, the extracted mode shapes are not scaled. Therefore, the mass change method which is a known technique for scaling is applied to scale the mode shapes.

2. WAVELET TRANSFORM

In this section a review of the continuous wavelet transform theory is given briefly. A detailed study can be found in [16 and 17]. A wavelet \( \psi (t) \) known also as the mother wavelet is a function which has two important features: first it must oscillate, second it must decay to zero. This means mathematically a wavelet \( \psi (t) \) with its Fourier transform \( \Psi (\omega) \) satisfies the following equation:

\[
0 < C_\psi = \int_{-\infty}^{\infty} |\Psi (\omega)|^2 \, d\omega < +\infty
\]  

(1)

This property is known as admissibility condition [17]. The wavelet transform of a signal \( x(t) \) is the inner product [11,17 and 18] in the Hilbert space as follows:

\[
W_\psi (s, \tau) = < x(t), \psi_s (t) > = \int_{-\infty}^{\infty} x(t) \psi^*_{s, \tau} (t) \, dt
\]

where \( * \) stands for the complex conjugate. \( \psi_{s, \tau} (t) \) is the son wavelet which is generated from the mother wavelet as:

\[
\psi_{s, \tau} (t) = \frac{1}{\sqrt{s}} \psi \left( \frac{t-\tau}{s} \right) \quad s > 0, \tau \in \Re
\]

(3)

where \( s \) is the dilatation or scale parameter which controls the width of the wavelet. \( \tau \) is the translation parameter localising the wavelet in the time domain. The wavelet decomposes a signal \( x(t) \) into the wavelet coefficients \( W_\psi (s, \tau) \) based on the son wavelets \( \psi_{s, \tau} (t) \).

Signal \( x(t) \) is required to decay to zero as \( t \to \pm \infty \) so that its wavelet transform in Eq. (2) exists. The inverse of the wavelet transform is given by:

\[
x(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_\psi (s, \tau) \psi_{s, \tau} (t) \, ds \, d\tau
\]

(4)

where \( C_\psi \) has been defined in Eq. (1).

Among the number of different wavelets, one of the most widely used in vibration analysis is the Morlet wavelet. The general form of Morlet function [18] is defined as:

\[
\psi(t) = B (e^{j\omega_0 t} - e^{-\omega_0^2 t^2 / 2}) e^{-t^2 / 2}
\]

(5)

where \( \omega_0 \) is the central wavelet frequency. In practice \( \omega_0 > 5 \), so that the Morlet wavelet satisfies the admissibility condition given in Eq. (1) [19]. In this work, a modified Morlet wavelet defined in [19], is used given by:

\[
\psi(t) = \frac{1}{\sqrt{2}} (e^{j\omega_0 t} - \frac{-\omega_0^2 N}{2} e^{-t^2 / 2 N}) e^{-t^2 / 2 N} \quad N > 0
\]

(6)
Parameter $N$ adjusts the Morlet wavelet so that the optimum resolution for decomposing the input signal in the time and frequency domain is obtained [19]. Although the Morlet wavelet is complex, its Fourier transform is real [19] and is given by:

$$\Psi(s \omega) = \sqrt{N \pi} \left( e^{-\frac{(s \omega - \omega_0)^2}{4}} - e^{-\frac{(s \omega + \omega_0)^2}{4}} \right)$$  \hspace{1cm} (7)

The first term in Eq. (7) is much greater than the second term when $\sqrt{N \omega_0} > 5$ which is the necessary condition for Morlet wavelet to satisfy the admissibility condition in Eq. (1). $\Psi(s \omega)$ is maximum when $s \omega = \omega_r$. Therefore the wavelet transform behaves as a band pass filter focused on the frequency $\omega$ which is determined by:

$$\omega = \frac{\omega_0}{s}$$  \hspace{1cm} (8)

By increasing $N$, the frequency resolution increases while the time resolution decreases. Therefore the optimum resolution both for time and frequency can be obtained by adjusting the parameter $N$ [19].

In this paper, in order to tune the parameter $N$ and evaluate the accuracy of the wavelet transform, the signal is regenerated using Eq. (4) and compared with the original time signal based on Root Mean Square (RMS) [20]:

$$E = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (S_i - RS_i)^2}$$  \hspace{1cm} (9)

where $E$ stands for the error for a given time interval based on the RMS concept, $n$ is the number of time samples, $S_i$ is the original time response at $i$th time sample and $RS_i$ is the regenerated time signal at $i$th time sample defined in Eq. (4). Therefore, the optimum value of parameter $N$ can be chosen to minimize the value of error parameter $E$.

### 3. Determination of Natural Frequencies and Damping Ratios

It is shown in [21] that the complex function of free response is:

$$y_i(t) = \sum_{r=1}^{n} A_r^i e^{-\zeta_r \omega_r t} e^{i(\omega_r t + \theta_r)}$$  \hspace{1cm} (10)

Eq. (10) is a good approximation of the analytical signal $x_i(t)$ providing that $y_i(t)$ is assumed to be asymptotic. The wavelet coefficients of the free vibration of system can be approximated using the asymptotic techniques [21] as:

$$W_\psi(s, \tau) = \sqrt{s} \sum_{r=1}^{n} A_r^i e^{-\zeta_r \omega_r \tau} e^{i(\omega_r \tau + \theta_r)} \psi^*(s \omega_r)$$  \hspace{1cm} (11)

The modulus of wavelet transform maximizes at a constant value of a scale parameter $s = s_r$. The parameter $s_r$ is associated with only one of the system modes when:

$$s_r = \frac{\omega_0}{\omega_{dr}}$$  \hspace{1cm} (12)

where $\omega_0$ is the central wavelet frequency of the $r$th mode. For the dilatation parameter $s = s_r$, only the $r$th mode gives a significant contribution to Eq. (11) and the effects of other modes are negligible. Therefore, for the $r$th mode, $\omega_\psi(s_r, \tau)$ can be approximated as:

$$W_\psi(s_r, \tau) = \sqrt{s_r} A_r^i e^{-\zeta_r \omega_r \tau} e^{i(\omega_r \tau + \theta_r)} \psi^*(s_r \omega_r)$$  \hspace{1cm} (13)

Consequently the natural frequency $\omega_r$ and damping ratios $\zeta_r$ can be determined as:
\[ \omega_r = \sqrt{\left( \frac{d}{d\tau} \ln|W_\psi(s_r, \tau)| \right)^2 + \left( \frac{d}{d\tau} (\text{Arg}(W_\psi(s_r, \tau))) \right)^2} \]  
(14)

\[ \zeta_r = \frac{-d}{d\tau} \ln|W_\psi(s_r, \tau)| \]  
(15)

4. NEW METHOD FOR MODE SHAPE IDENTIFICATION

For a MDOF system with proportional viscous damping, the mode shapes are identical to those of the undamped system [1]. The response for each degree of freedom for undamped system can be calculated by:

\[ y_i(t) = \sum_{r=1}^{n} A_r \cos(\omega_r t + \theta_r) \]  
(16)

when:

\[ \zeta_r = 0 \]  
(17)

Eq. (16) can be formulated in the matrix form as:

\[ Y(t) = \sum_{r=1}^{n} A_r \cos(\omega_r t + \theta_r) \]  
(18)

where \( A_r \) is the unscaled mode shapes of system. If the system is excited by an impact force, the initial conditions are:

\[ Y(0) = 0 \]  
(19)

\[ V(0) = V_0 \]  
(20)

Inserting Eq. (19) into Eq. (18); gives:

\[ \sum_{r=1}^{n} A_r \cos \theta_r = 0 \]  
(21)

As the mode shape vectors are linearly independent, it is concluded that:

\[ \cos \theta_r = 0 \]  
(22)

which results in:

\[ \theta_r = 2k\pi + \frac{\pi}{2} \quad k = 0, \pm 1, \pm 2, \ldots \]  
(23)

By inserting Eq. (23) in to Eq. (18), the following equation is obtained:

\[ Y(t) = \sum_{r=1}^{n} A_r \sin \omega_r t \]  
(24)

In which the elements of \( A_r \) can also be negative to compensate the negative sign of \( \cos(\omega_r t + \theta_r) \) when \( \theta_r = 2k\pi + \frac{\pi}{2}, k = 0, \pm 1, \pm 2, \ldots \)

Eq. (24) can be rewritten as:

\[
\begin{pmatrix}
    y_1(t) \\
    y_2(t) \\
    \vdots \\
    y_n (t)
\end{pmatrix}
= \begin{pmatrix}
    A_1^i \\
    A_1^j \\
    \vdots \\
    A_1^n
\end{pmatrix} \sin \omega_1 t + \begin{pmatrix}
    A_2^i \\
    A_2^j \\
    \vdots \\
    A_2^n
\end{pmatrix} \sin \omega_2 t + \ldots
\]  
(25)

where \( A_r^m \) is the \( m \)th element of the \( r \)th mode shape. Eq. (25) can be rearranged as:
The time signals at different DOFs of vibrating system can be obtained by measuring the response at each DOF from the impact test.

Eq. (26) can be written for different times: \( t_1, t_2, t_3, \ldots, t_n \). These equations can be rearranged in matrix form as follows:

\[
\begin{bmatrix}
  y_1(t_1) & y_2(t_1) & \cdots & y_n(t_1) \\
  y_2(t_2) & y_2(t_2) & \cdots & y_n(t_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  y_n(t_1) & y_2(t_1) & \cdots & y_n(t_n)
\end{bmatrix} =
\begin{bmatrix}
  A_1^1 & A_1^2 & \cdots & A_1^n \\
  A_2^1 & A_2^2 & \cdots & A_2^n \\
  \vdots & \vdots & \ddots & \vdots \\
  A_n^1 & A_n^2 & \cdots & A_n^n \\
\end{bmatrix} \begin{bmatrix}
  M & M & \cdots & M \\
  M & M & \cdots & M \\
  \vdots & \vdots & \ddots & \vdots \\
  M & M & \cdots & M \\
\end{bmatrix}
\begin{bmatrix}
  \sin \omega_1 t_1 & \sin \omega_2 t_1 & \cdots & \sin \omega_n t_1 \\
  \sin \omega_1 t_2 & \sin \omega_2 t_2 & \cdots & \sin \omega_n t_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  \sin \omega_1 t_s & \sin \omega_2 t_s & \cdots & \sin \omega_n t_s
\end{bmatrix}
\]

The matrix of mode shapes can be derived as:

\[
\begin{bmatrix}
  A_1^1 & A_1^2 & \cdots & A_1^n \\
  A_2^1 & A_2^2 & \cdots & A_2^n \\
  \vdots & \vdots & \ddots & \vdots \\
  A_n^1 & A_n^2 & \cdots & A_n^n \\
\end{bmatrix} =
\begin{bmatrix}
  \sin \omega_1 t_1 & \sin \omega_2 t_1 & \cdots & \sin \omega_n t_1 \\
  \sin \omega_1 t_2 & \sin \omega_2 t_2 & \cdots & \sin \omega_n t_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  \sin \omega_1 t_s & \sin \omega_2 t_s & \cdots & \sin \omega_n t_s
\end{bmatrix}^+ \begin{bmatrix}
  y_1(t_1) & y_2(t_1) & \cdots & y_n(t_1) \\
  y_2(t_2) & y_2(t_2) & \cdots & y_n(t_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  y_n(t_1) & y_2(t_1) & \cdots & y_n(t_n)
\end{bmatrix}
\]

Where “+” refers to the pseudo-inverse of a matrix. Each row of the matrix on the left hand side of Eq. (26) is related to one of the mode shapes of structure. However, as these mode shapes are extracted only from the response data, they are not scaled and are required to be scaled using one of the known scaling methods such as mass change method.

There are three sources of error in determination of mode shapes using the proposed method:

- Noise in the response signal.
- Inaccuracy in the estimated natural frequencies from wavelet analysis due to noise or other sources of error.
- The truncation error due to not taking to account all the natural frequencies of the structure.

4. **Experimental Case Study**

In order to validate the proposed mode shape identification technique in a real test, the method was applied to a clamped-clamped beam (Fig.1). The beam was made of steel and had dimensions 6×4×700 mm. The beam was discretized to nine elements. The modal parameters of beam were extracted using two methods: first, conventional hammer test and second, the wavelet transform. The unscaled mode shapes obtained from the proposed method were scaled using the mass change method.
5.1 HAMMER TEST

The conventional hammer modal testing was conducted allowing the estimation of the natural frequencies, damping ratios and scaled mode shapes of beam. Eight accelerometers type DJB A/120VT were mounted on points 2-9 of beam (Fig. 1).

Figure 1: Hammer modal testing on the clamped-clamped beam

Point 3 was selected for the excitation based on the theoretical relations for choosing the best point for excitation as given in [22]. The beam was excited by a hammer type BK8202 and the response and force signals were measured using an Analyzer type B&K 3560 D. The frequency response functions were extracted using Pulse 8 software (Fig. 2). The MODENT module of ICATS software was used to obtain the first five natural frequencies, Damping ratios and scaled mode shapes of the beam. The natural frequencies from the hammer test are given in Table 1.

Figure 2: FRFs from the hammer test

Table 1: First five natural frequencies of the clamped-clamped beam from conventional hammer testing and wavelet method

<table>
<thead>
<tr>
<th>Mode #</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural frequencies from hammer test (Hz)</td>
<td>54.06</td>
<td>146.59</td>
<td>291.31</td>
<td>477.34</td>
<td>712.07</td>
</tr>
<tr>
<td>Natural frequencies from wavelet method (Hz)</td>
<td>54.12</td>
<td>146.49</td>
<td>288.89</td>
<td>476.65</td>
<td>716.85</td>
</tr>
</tbody>
</table>

5.2 MODAL PARAMETERS IDENTIFICATION USING WAVELET TRANSFORM

The measured responses of beam (from the pervious hammer test) were used to estimate the natural frequencies and damping ratios using the wavelet procedure as explained in section 3. The obtained natural frequencies are compared with those of the hammer test in Table 1 showing that the extracted natural frequencies from the wavelet transform are close to those of the conventional hammer test. On the other hand the wavelet transform was not successful in estimating the damping ratios due to noise. Then the mode shapes were estimated using obtained natural frequencies from wavelet method and measured responses based on the procedure as explained in section 4.

The obtained mode shapes from the wavelet transform are compared with those of the hammer test based on the MAC criterion in Fig. 3, showing that the extracted mode shapes are in complete correlation with the hammer mode shapes although they are not scaled. It should be noted that in this case only the first five natural frequencies are considered. Although not all of the natural frequencies are taken in to account, the extracted mode shapes are accurate.
5.2 MASS CHANGE METHOD

The mass change method was applied to scale the unscaled mode shapes from the proposed method [23]. Eight 11.5 gram masses were added to points 2-9 on the beam. The experimental setup with added masses is shown in Fig. 4.

The hammer test was conducted on the beam and the new natural frequencies and mode shapes were obtained from the free decay response using the wavelet transform. The mode shapes were scaled using mass change method. The scaled mode shapes from the wavelet transform are compared with the mode shapes from hammer test in Fig. 5 showing that the proposed method can estimate the scaled mode shapes.

9. CONCLUSIONS

A procedure for identifying the modal parameters in time domain using the modified Morlet wavelet function is presented. The free response of the vibrating system subjected to an impulsive force is used as the input data. A method is proposed for extracting the unscaled mode shapes from the response signals and the extracted natural frequencies from wavelet. The proposed procedure has been examined by a real test on a beam demonstrating that the procedure is accurate in the estimating of natural frequencies and mode shapes and is affected by noise in identifying the damping ratios.
11. REFERENCES