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A new method for solving of a backward stochastic differential equations by using a basic functions

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Abstract
In this paper, we purpose a method for numerical solution of a backward stochastic differential equations driven by standard Brownian motion as follows:

$$\begin{cases} dX(s) = f(X(s))ds + g(X(s))dB(s), & s \in [0,T), \\ X(T) = p. \end{cases}$$

The method is stated by using the basic functions based on the block pulse functions. Finally, we show the method has a good degree of accuracy by using some examples.

Keywords and phrases: Block pulse functions; Backward stochastic differential equations.

1. Introduction

Let $B(t),\ t \geq 0$ be the standard Brownian that be a martingale process and Gaussian (see [2]). In process have many applications in mathematical finance, biology, medical, social, scienes, etc (see [1]).
In this artical, we consider the stochastic differential equation

$$\begin{cases} dX(s) = f(X(s))ds + g(X(s))dB(s), & s \in [0,T), \\ X(T) = p. \end{cases}$$

or

$$X(t) = p + \int_t^T f(X(s))ds + \int_t^T g(X(s))dB(s) \qquad 0 \leq t < T. \tag{1.1}$$

where $X(S)$ be a stochastic process and unknown and $f,\ g : [0,T) \to \mathbb{R}$.

This paper is organized as follows. In section 2, we state a essential theorem and the basic properties of the block pulse functions. Then, we introduce the concept of the stochastic integration operational matrix. In section 3, we solve Eq. (1) by using the stochastic integration operational matrix and collocation method. In section 4, we examine The proposed method with an example. Finally, Section 5, we give a brief conclusion.

2. Preliminaries

Theorem 2.1. *Let $f(x)$ be twice continuously differentiable function on $\mathbb{R}$, then for all $t \in (0,T]$*

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt.$$

Proof. See [1].
Now, we state the properties of BPFs, for many details see [3, 4, 5].

2.1. **Definition.** We define the m-set of BPFs as

$$\phi_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih, \quad i = 1, ..., m, \\ 0 & otherwise. \end{cases}$$

and $h = T/m$. The elementary properties of BPFs are as follows

1. The BPFs are disjointed with each other in the interval $t \in [0, T)$

$$\Phi_i(t)\Phi_j(t) = \delta_{ij}\Phi_i(t).$$

2. THE BPFs are orthogonal with each other in the interval $t \in [0, T]$

$$\int_0^T \Phi_i(t)\Phi_j(t)dt = h\delta_{ij}, \ i, j = 1, 2.$$

3. For every $f \in L^2([0, T))$,when m approaches to the infinity, Parseval's identity holds:

$$\int_0^T f^2(t)dt \cong \sum_{i=1}^{\infty} f_i^2 ||\Phi_i(t)||^2,$$

where

$$f_i = 1/h \int_0^T f(t)\Phi_i(t)dt.$$

Vector form: consider the first m terms of BPFs and write them concisely as m-vector,

$$\Phi(t) = (\Phi_1(t), \Phi_2(t), \cdots, \Phi_m(t))^T, t \in [0, T).$$

The above representation and disjointedness property follows:

$$\phi_i(t)\phi_i^T(t) = \begin{cases} \phi_i(t) & (i-1)h \leq t < ih, \quad i = 1, ..., m, \\ 0 & otherwise. \end{cases}$$

Furthermore, we have

$$\Phi^T(t)\Phi(t) = 1,$$

and

$$\Phi(t)\Phi^T(t)V = \hat{V}\Phi(t),$$

where, V is an m-vector and $\hat{V} = diag(V)$. Moreover, it can be clearly concluded that for every $\times mm$ matrix

$$\Phi^T(t)A\Phi(t) = \hat{A}\Phi(t),$$

where, $\hat{A^T}$ is an m-vector with elements equal to the diagonal entries of matrix A.

2.2. **Functions approximation.** An arbitrary real bounded function $f(t)$, which is square integrable in the interval $t \in [0, T)$, can be expanded into a block pulse series in the sense of minimizing the mean square error between $f(t)$ and its approximation

$$f(t) \simeq \hat{f_m}(t) = \sum_{i=1}^{m} f_i\Phi_i(t),$$

where f_i is the block pulse coefficient with respect to the ith BPFs $\Phi_i(t)$. In the vector from we have,

$$f(t) = \hat{f_m}(t) = F^T\Phi(t) = \Phi^T(t)F,$$

where

$$F = (f_1, f_2, ..., f_m)^T.$$

Let $K(t,s)$ is a function of two variables in $L^2([0,T_1)*[0,T_2))$. It can be similarly expanded with respect to BPFs as

$$K(t,s)=\Phi^T(t)K\Psi(s),$$

where, $\Psi(s)$ and $\Phi(t)$ are m_1, m_2 dimensional BPFs vectors respectively ,and

$$K=(K_(ij)) \quad ,i=1,2,\cdots,m_1, j=1,2,\cdots,m_2,$$

is the $m_1.m_2$ block pulse coefficient matrix with

$$K(ij)=\frac{1}{h_1h_2}\int_0^{T_1}\int_0^{T_2}K(t,s)\Phi_i(t)\Psi_j(s)dtds.$$

where, $h_1=T_1/M_1$, $h_2=T_2/m_2$. For convenience , we put $m_1=m_2=m$. Integration operational The Lebesgue integral and Itô integral of BPFs can be computed [4] as follows:

$$\int_0^t \Phi(s)ds=P\Phi(t),$$

and

$$\int_0^t dB(s)=P_s\Phi(t),$$

where operational matrix P and P_s are given by

$$P=\frac{h}{2}\begin{pmatrix}1&2&2&\dots&2\\0&1&2&\dots&2\\0&0&1&\dots&2\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\end{pmatrix}_{m\times m},$$

and

$$P_s=\begin{pmatrix}B(\frac{h}{2})&B(h)&B(h)&\dots&B(h)\\0&B(\frac{3h}{2})-B(h)&B(2h)-B(h)&\dots&B(2h)-B(h)\\0&0&B(\frac{5h}{2})-B(2h)&\dots&B(3h)-B(2h)\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&B(\frac{(2m-1)h}{2})-B((m-1)h)\end{pmatrix}_{m\times m},$$

so, we can write

$$\int_0^t f(s)ds=\int_0^t F^T\Phi(s)ds=F^TP\Phi(t).$$

and

$$\int_0^t f(s)dB(s)=\int_0^t F^T\Phi(s)dB(s)=F^TP_s\Phi(t).$$

3. Numerical Solution

Let

$$\begin{cases} f(X(s)) = r(s), \\ g(X(s)) = p(s), \end{cases} \tag{3.1}$$

by substituting the relation (2) in Eq. (1), we obtain:

$$X(t) = p + \int_t^T r(s)ds + \int_t^T p(s)dB(s). \tag{3.2}$$

Let

$$r = T - s, \tag{3.3}$$

by using the relation (4) and Theorem (2.1), we obtain

$$X(t) = p - \int_{T-t}^0 r(n)dn - \int_{T-t}^0 p(n)dB(n). \tag{3.4}$$

Now, by using properties of the BPFs, we have

$$\begin{cases} r(s) = r^T\Phi(s) = \Phi^T(s)r, \\ p(s) = p^T\Phi(s) = \Phi^T(s)p, \end{cases} \tag{3.5}$$

by using the relation (6), we get

$$X(t) = p - \int_{T-t}^0 r^T\Phi(n)dn - \int_{T-t}^0 p^T\Phi(n)dB(n), \tag{3.6}$$

or,

$$X(t) = p + r^T P\Phi(T-t) + p^T P_s\Phi(T-t). \tag{3.7}$$

By substituting the relation (8) into (2) and collocation technique in m nodes $T - t_j = \frac{j}{m+1}$ and $j = 1, \ldots, m$, we get

$$\begin{cases} r(t_j) = f(p + r^T P\Phi(T-t_j) + p^T P_s\Phi(T-t_j)), \\ p(t_j) = g(p + r^T P\Phi(T-t_j) + p^T P_s\Phi(T-t_j)), \end{cases} \tag{3.8}$$

or,

$$\begin{cases} r^T\Phi(t_j) = f(p + r^T P\Phi(T-t_j) + p^T P_s\Phi(T-t_j)), \\ p^T\Phi(t_j) = g(p + r^T P\Phi(T-t_j) + p^T P_s\Phi(T-t_j)), \end{cases} \tag{3.9}$$

After solving the relation (10), we obtain

$$X(t) = p + r^T P\Phi(T-t) + p^T P_s\Phi(T-t). \tag{3.10}$$

4. Numerical examples

Example 4.1. *Let*

$$\begin{cases} dX(s) = \frac{1-X(s)}{1-s}ds + dB(s), \ \ s \in [0,1), \\ X(1) = 1, \end{cases} \tag{4.1}$$

With exact solution $X(t) = t + (1-t)\int_0^t \frac{dB(s)}{1-s}$ and $X(t)$ is an unknown stochastic process defined on the probability spaces (Ω, F, P). The numerical results have been shown in Tables (1). In Table1, n is the number of iterations, $\overline{x}_E$ is error mean and s_E is standard deviation of error.

Table 2: Mean, Standard deviation and Confidence interval for numerical solution mean (T=0.25, m=10)

s	$\overline{x}_E$	s_E	%95 Confidence interval for mean of E	
			Lower	*Upper*
0.05	1.7605×10^{-4}	1.3426×10^{-4}	1.1721×10^{-4}	2.3489×10^{-4}
0.1	3.6390×10^{-4}	3.0606×10^{-4}	2.2976×10^{-4}	4.9804×10^{-4}
0.15	5.6640×10^{-4}	5.2972×10^{-4}	3.3423×10^{-4}	7.9857×10^{-4}
0.2	6.4855×10^{-4}	5.6226×10^{-4}	4.0213×10^{-4}	8.9497×10^{-4}

5. Conclusion

In this paper We suggest a method for solving the stochastic back ward differential equation driven standard Brownian motion by using stochastic operational matrix based on the block pulse functions in combination with collocation method. Also, good approximate of method is shown in an example.

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