3rd Conference on Financial Mathematics & Applications.
30, 31 January 2013, Semnan University, Semnan, Iran, pp. xxx-xxx

A numerical method for portfolio selection based on Markov chain approximation

Z. Nikoueinezhad* **B. Kafash**† **A. Delavarkhalafi**‡
*,†,‡ *Faculty of Mathematics, Yazd University, Yazd, Iran.*
†*Emam Javad Higher Education Institute, yazd, Iran.*
E-mail: Nikoueinezhad@yahoo.com,
E-mail: Bkafash@stu.yazduni.ac.ir, Delavarkh@yazduni.ac.ir

Abstract
In this paper, A portfolio selection problem is approximated by a Markov chain which is modulated by a continuous-time, finite-state, Markov chain. The main ingredient of the Markov chain approximation is to approximate the wealth process and utility function of original utility optimization problem by a controlled Markov chain. under the convergence of the approximation scheme, Policy iteration methods as to obtain the optimal controls. A numerical example is provided to illustrate the reability of the algorithm.

Keywords and phrases: Numerical method, Portfolio selection, Stochastic optimal control, Markov chain approximation.

1. Introduction

Portfolio selection is one of important problems in mathematical finance, which was first explored by Harry Markowitz. Portfolio selection prblem which attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently minimize risk for a given level of expected return, by carefully choosing the proportions of various assets. Although is widely used in practice in the financial industry and several of its creators won a Nobel memorial prize for the this theory in recent years the portfolio selection have been widely challenged by fields such as behavioral economics.
In our model, We consider a continuous-time financial model consisting of two primitive assets, namely, a money market account and a stock. In the real market, investors cannot put too much money in risky assets for the sake of risk management. For example, there is a golden rule: "never borrow money to do risky investment". That is, there is a natural constraint on the portfolio so that the total weight of the risky assets should be no more than 1.
Although, In generally Some of portfolio selection problems can only be described by the associated HJB equations, which are difficult to solve. the associated HJB equation is very difficult to solve explicitly, and generally the solution cannot be deduced in an explicit form. In this paper, rather than focusing on analytic solutions, we present an attempt to solve the problem numerically.
We consider a continuous-time financial model consisting of two primitive assets, namely, money market account and stock. we use the expected utility as the objective of the portfolio

selection. our target is to maximize the expected utility of the terminal wealth by choosing proper portfolios. In a real market, most investors, especially fund managers, will never invest too heavily in any risky asset for the sake of risk control [1, 2, 3].
This work focuses on the development of approximation algorithms for the portfolio optimization problems. The main technique used is the Markov chain approximation methods. At first, the wealth process and utility function of original utility optimization problem is approximated by a controlled Markov chain. we carry out the following steps.
(i) We construct a controlled Markov chain.
(ii) We show that a suitably interpolated process of the Markov chain converges to the desired limit, leading to the system equation, the cost, and the value functions. To construct the controlled Markov chain, it is essential to obtain the right transition probability matrix. This chain needs to be "consistent" with the diffusion characteristics, which will be discussed later. We need only come up with a way of constructing a "right" controlled Markov chain, and proving the limit to be the desired one; how we construct the chain is not of concern. Policy iteration methods, which are widely used in the control literature, are adopted to obtain optimal control

2. PROBLEM STATEMENT

We consider a continuous-time financial model consisting of two primitive assets, namely, a money market account and a stock. These assets are assumed to be tradable continuously on a fixed time horizon $\tau = [0, T]$. evolution of the risk-free asset follows:

$$dS_0(t) = S_0(t)r(t)dt \tag{2.1}$$

The evolution of the price process of the stock(Risky asset) follows a GBM:

$$dS_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dW(t)], \qquad S_1(0) = s_1 > 0 : \tag{2.2}$$

In the sequel, we describe the evolution of the wealth process of an investor who allocates his/her wealth between the money market account and the stock. Let $\pi(t)$ denote the proportion of the total wealth invested in the stock at time $t \in T$.
Let $X(t) = X^{\pi}(t)$ denote the total wealth of the portfolio π at time t. Then, the evolution of the wealth process $X = \{X(t)\}$ over time is governed by:

$$dX(t) = X(t)[r(t) + \pi(t)(\mu(t) - r(t))]dt + X(t)\pi(t)\sigma(t)dW(t), \qquad X(0) = x. \tag{2.3}$$

Our goal is to find a portfolio process $\{\pi(t)\}$ which maximizes the utility of the terminal wealth.
The portfolio optimization problem can be written as:

$$Maximize E^{\pi}_{t,x} U(X(T))$$

$$dX(t) = X(t)\{[r(t) + \pi(t)(\mu(t) - r(t))]dt + \pi(t)\sigma(t)dW(t), \} \qquad X(0) = x. \tag{2.4}$$

Define $V(x, t)$ as the optimal value of the problem. That is,

$$V(x, t) = \sup_{\pi} E_{t,x} U(X(T)).$$

the associated HJB equation of portfolio selection is:

$$V_t + xV_x r(t) + \sup_{\pi\in\Pi}\{xV_x\pi(t)(\mu(t)-r(t)) + \frac{1}{2}x^2V_{xx} \mid \pi(t)\sigma(t) \mid^2\} = 0,$$
$$V(T,x) = U(x). \tag{2.5}$$

3. Driving method

This section focuses on the development of approximation algorithms for the portfolio optimization problems. we carry out the following steps. (i) We construct a controlled Markov chain. (ii) We show that a suitably interpolated process of the Markov chain converges to the desired limit, leading to the system equation, the cost, and the value functions. To construct the controlled Markov chain, it is essential to obtain the right transition probability matrix. This chain needs to be "consistent" with the diffusion process. Policy iteration methods, which are widely used in the control literature, are adopted for the actual computation. There is no need to discretize the HJB equation satisfied by the value function.
There are many ways of constructing the Markov chain. Because the value function depends on both the state x and the time variable t , two stepsizes are needed. We use $h > 0$ as the stepsize for the state and $\delta > 0$ as the stepsize for time, for the given $T > 0$. Let $\{\xi_n^h, n < \infty\}$ be a controlled discrete-time Markov chain on the discrete state space S^h with time-dependent transition probabilities from a state x to another state y, denoted by $P^{h,\delta}(x,y|\alpha)$ for $\alpha \in \Pi$. We use $\pi_n^{h,\delta}$ to denote the random variable that is the control action for the chain at discrete time n . To figure out the form of $\pi_n^{h,\delta}$ we define a finite difference approximation to find transition probabilities from approximation Markov chain that is "consistent" to wealth process.
For each feasible portfolio $\pi_t = \pi(t,X_t)$ we can define $U(t,x,\pi) = E^{\pi}_{t,x}U(X(T))$.Then $U(t,x,\pi)$ formally satisfies the partial differential equation:

$$\begin{aligned} U_t(t,x,\pi) &+ xU_x(t,x,\pi)r(t) \\ &+ \{xU_x(t,x,\pi)\pi(t)(\mu(t)-r(t)) + \frac{1}{2}x^2U_{xx}(t,x,\pi) \mid \pi(t)\sigma(t) \mid^2\} = 0, \end{aligned} \tag{3.1}$$

We use finite difference method to find transition probabilities from approximation Markov chain that is "consistent" to wealth process. Define the approximation to the first and the second partial derivatives of $g(.)$ as

$$\begin{aligned} g_t(x,t) &= \frac{g(x,t+\delta)-g(x,t)}{\delta}, \\ g_x(x,t) &= \frac{g(x+h,t+\delta)-g(x,t+\delta)}{h}, \qquad for\, x[r+\alpha' B] \geq 0, \\ g_x(x,t) &= \frac{g(x,t+\delta)-g(x-h,t+\delta)}{h}, \qquad for\, x[r+\alpha' B] < 0, \\ g_{xx}(x,t) &= \frac{g(x+h,t+\delta)-2g(x,t+\delta)+g(x-h,t+\delta)}{h^2}. \end{aligned} \tag{3.2}$$

Use the discretization (3.2) and substitute in (3.1) ,therefore transition probability will be:

$$P^{h,\delta}(x,x,n\delta \mid \alpha) = 1 - \rho(x,n\delta,\alpha)\frac{\delta}{h^2} - [x(r(n\delta)) + \alpha(x,n\delta)B(n\delta)]\frac{\delta}{h}$$
$$P^{h,\delta}(x,x+h,n\delta \mid \alpha) = \rho(x,n\delta,\alpha)\frac{\delta}{h^2} + [x(r(n\delta)) + \alpha(x,n\delta)B(n\delta)]^+\frac{\delta}{h},$$
$$P^{h,\delta}(x,x+h,n\delta \mid \alpha) = \rho(x,n\delta,\alpha)\frac{\delta}{h^2} + [x(r(n\delta)) + \alpha(x,n\delta)B(n\delta)]^-\frac{\delta}{h}. \qquad (3.3)$$

By choosing δ and h appropriately, we can make $P^{h,\delta\alpha}(x,x,n\delta \mid \alpha) > 0$ is nonnegative and summation of transition probability equals to 1.
It will be showed that sequence $\{\xi_n^{h,\delta} : n < \infty\}$ with obtained transition probability is consistent. This means

$$E_{x,n}^{\alpha,h,\delta} \triangle \xi_n^{h,\delta} = x[r(n\delta)) + \alpha(x,n\delta)B(n\delta)]\delta + o(\delta),$$
$$Var_{x,n}^{\alpha,h,\delta} \triangle \xi_n^{h,\delta} = [x\alpha\sigma(n\delta)]^2\delta + o(\delta). \qquad (3.4)$$

With the approximating Markov chain constructed above, we can obtain an approximation of the utility function defined in (2.4). We have

$$U^{h,\delta}(x,n\delta,\pi^{h,\delta}) = \sum_y P^{h,\delta}(x,y \mid \pi^{h,\delta}(x,n\delta))U^{h,\delta}(y,n\delta+\delta,\pi^{h,\delta}). \qquad (3.5)$$

Therfore,

$$V^{h,\delta}(t,x) = \sup_{\pi^{h,\delta}} U^{h,\delta}(x,n\delta,\pi^{h,\delta}). \qquad (3.6)$$

Our aim is to show that $V^{h,\delta}(t,x)$ converges to the value $V(t,x)$ as $h,\delta \to 0$. Again, in the actual computation, the optimal control is obtained using policy improvement methods.

3.1. Approximation in Policy Space.

Theorem 3.1. *Assume that there is a unique solution to (3.1), and it is the infimum of the cost functions over all time independent feedback controls. Let $\pi_0(.)$ be an admissible feedback control such that the cost $U(\pi_0)$ is bounded. For $n \geq 1$, define the sequence of feedback controls $\pi_n(.)$ and costs $U(\pi_n)$ recursively together with the formula*

$$\pi_{n+1}(x) = \arg\min_{\alpha\in\mathcal{U}}[\sum r(x,y|\alpha)U(y,\pi_n). \qquad (3.7)$$

Then $U(\pi_n) \to V$.

Proof. See [5] □

4. Application to the mean-variance problem

In the mean-variance problem, the optimal solution is often obtained by a quadratic utility function of the form

$$u(x) = -\frac{(x-\lambda)^2}{2},$$

For simplicity, the discussion is confined to a case of one risky assets in the market with interest rate $r = 0.03$, and return rates 0.15 and $\sigma(t) = 0.1, B(t) = 0.12$.

Therfore, We have

$$dS_0(t) = S_0(t)0.03dt$$

$$dS_1(t) = S_1(t)[0.15dt + 0.1dW(t)], \qquad S_1(0) = s_1 > 0;$$

The wealth of investor will be

$$dX(t) = X(t)[0.03 + \pi(t)0.12]dt + X(t)\pi(t)0.1dW(t), \qquad X(0) = x. \tag{4.1}$$

The goal of investor is to find a portfolio that maximize the expected utility function of finanl invest.
Fixing $\delta = 0.04, h = 0.2$, we obtain the computation results depicted in Figs as follows. In these figures, we take the wealth x at time.

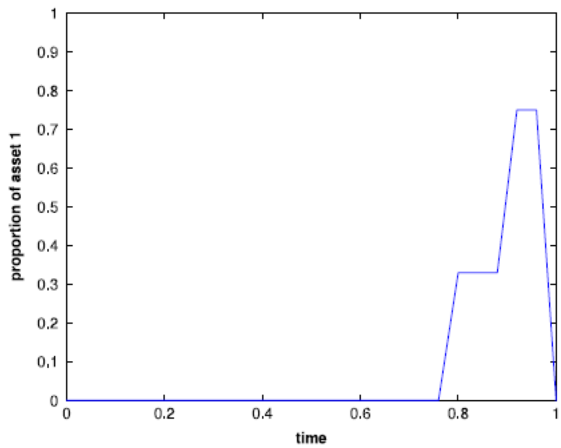

FIGURE 1. Proportion of asset 1 vs. time with wealth $x = 0.2$

References

1. FLEMING, W.H., AND PANG, T., *An application of stochastic control theory to financial economics*, SIAM J. Control Optim. 43 (2004) 502-531.
2. FLEMING, W.H., AND RISHEL, C.J., *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, NY, 1975.
3. FLEMING, W.H., AND SONER, H.M., *Controlled Markov Processes and Viscosity Solutions*, Springer, 2006.
4. HANSON, F. B. , *Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis and Computation*, University of Illinois Chicago, Illinois, USA, 2007.
5. KUSHNER, H.J., AND DUPUIS, P.G., *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd Edition, Springer-Verlag, New York, NY, 2001.