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A Survey on exact analytical and numerical solutions of some S.D.E.s based on martingale approach and changing variable method

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Abstract
In this paper, we decide to represent analytical and numerical solutions for stochastic differential equations, specially reputed and famous equations in pricing and investment rate models. By making martingale process from an arbitrary process in $L^2(\mathbb{R})$ space, we infer equations just with stochastic part (drift free). This method could be done by Ito product formula on initial process and an appropriate martingale process, then we compare simulating method of arising this new equation with other simulating method like as E.M. and Milstein. Another suitable method is converting S.D.E.s to O.D.E.s whom we try to omit diffusion part of stochastic equation. Afterwards, it could be solved by different numerical methods like as Runge-kutta from fourth order. In this paper, we solve well known equations such as Gampertz diffusion and logistic diffusion by this method. Another powerful one is change of variable method whom we could analysis and survey a well known group of stochastic equations like as special case of squared radial Langevin process, Cox-Ingersoll-Ross model and Ornstein-Uhlenbeck process. For numerical solution of these stochastic equations, we could apply wiener chaos expansion method whom we have described in other paper.

Keywords and phrases: Numerical solution, Ito formula, Stochastic equations, Wiener chaos expansion.

1. Introduction

As we know, analytical solution of partial and ordinary differential equations has been custom since long time ago. This kind of solutions are so important especially in physics and engineering [2, 4, 6, 9]. But most of equations don't have exact solution and even a limited number of these equations, for instance in classical form, have implicit solutions, although analytical methods and solutions could be so great and principle in some cases that we decide to compare and analyze different solutions and their errors with analytical solution[10, 11, 12]. Therefore, various numerical methods could be utilized in most differential equations [2, 5, 7]. Also about stochastic differential equations (S.D.E.s), this issue is correct and different numerical methods like $MontCarlo$ simulation, finite elements and finite differences are usual in finding numerical solutions of S.D.E. In this paper, we decide to represent analytical and numerical methods for stochastic differential equations, specially reputed and famous equations in pricing and investment rate models with various examples about them which we have found in various papers by making martingale process from an arbitrary process in $L^2(\mathbb{R})$ space [1, 8]. We infer equations just with diffusion part (drift free). This method could be done by *itô* product formula on initial process and an appropriate martingale process, then we compare simulating method of arising this new equation with other simulating methods like as $E.M.$ and $Milstein$ [1, 3]. Another suitable method is converting S.D.E.s to O.D.E.s whom we try to omit diffusion part of stochastic equation. Afterwards, it could be solved by different numerical methods like as $Rung-kutta$ from fourth order. In this paper, we solve well known equations such as *Gampertz* Diffusion and *Logistic* Diffusion by this method. Another powerful one is change of variable method whom we could analysis and survey a well known group of stochastic equations like as special case of squared radial *Langevin* process, $Cox-Ingersoll-Ross$ model and $Ornstein-Uhlenbeck$ process. A numerical solution of stochastic differential equations that we apply wiener chaos expansion method whom we have described in other paper.

2. Making Martingale Process

In this section, we intend under some conditions make a martingale process from a random one in $L^2(\mathbb{R}\times[0,T])$, which T, is called maturity time. We consider a $d-$dimensional wiener process W_t, such that $t\in[0,T]$ on the probability measure space (Ω, F, P, F_t) that F_t is a filtration of $\sigma-$Algebra F defined on space Ω. We define the $Exponential\ Martingale$ associated to $\lambda(t)$, that it could be a $d-$dimensional process [1].

$$Z_t^\lambda = \exp\left(\int_0^t \lambda_s dw_s - \frac{1}{2}\int_0^t |\lambda_s|^2 ds\right), \tag{2.1}$$

it could be indicated by itô formula that:

$$dZ_t^\lambda = \lambda Z_t^\lambda . dW \tag{2.2}$$

As we know, for every nonnegative random variable φ, which is $P-$ measurable, then

$$Q(A) := \int_A \varphi dP,$$

define a measure on space $(\Omega, \mathcal{F})$ which it is shown with $\varphi = \frac{dQ}{dP}$. A result on the change of probability measure for conditional expectations, is given as follows:

Theorem 2.1. ***(Bayes' formula)*** *Let P, Q be probability measures on space $(\Omega, \mathcal{F})$ defined as above. If $X \in \mathbb{L}^1(\Omega, Q)$ and $\mathcal{G}$, is a sub$-\sigma-$algebra of F and we set $\varphi = \frac{dQ}{dP}$ defined as before, then we have* [1]

$$E^Q[X|\mathcal{G}_t] = \frac{E^P[X\varphi|\mathcal{G}_t]}{E^P[\varphi|\mathcal{G}_t]} \ (t \in [0, T]). \tag{2.3}$$

If we define the new measure Q on space (Ω, P), by

$$Q(A) = \int_A \frac{dQ}{dP} dP \ , \ \frac{dQ}{dP} = Z_t^\lambda$$

that is named the $Radon - Nikodem$ density, so we have the following result:

Lemma 2.2. *If $E^p[X|F_t]$ and $E^Q[X|F_t]$ are conditional expectation respect to measures P, Q then*

$$E^Q[X|F_t] = \frac{E^P[XZ_t^\lambda|F_t]}{E^P[Z_t^\lambda|F_t]} \ (t \in [0, T]). \tag{2.4}$$

Finally, we could get the following result which is so important in next our conclusions.

$$\mu_t \text{ is Q-martingale} \Leftrightarrow \mu_t Z_t^\lambda \text{is P-martingale} \tag{2.5}$$

thus if μ_t is Q-martingale:

$$E^Q[\mu_t] = E^P[\mu_t Z_t^\lambda] = \mu_s Z_s^\lambda = \mu_0.$$

Of course, according to (2.5), we have

$$X_t \text{ is P-martingale} \Leftrightarrow X_t (Z_t^\lambda)^{-1} \text{is Q-martingale} \tag{2.6}$$

such that $(Z_t^\lambda)^{-1} = \exp(-\int_0^t \lambda_s.dW_s + \frac{1}{2}\int_0^t |\lambda_s|^2 ds)$.
Special case of change of measure **(Girsanov-Cameron-Martin Theorem)**, that $Radon - Nikodem$ density of new measure is exponential martingale, is represented as follows:

Theorem 2.3. ***(Girsanov Theorem)*** *Suppose* $dX^P = \mu(X,t)dt + \sigma(X,t)dW^P$*, that* W^P *is* $P-$*Brownian motion and*

$$Q = \int \frac{dQ}{dP} dP = \int_{-\infty}^{\infty} Z_t^\lambda dP.$$

Thus, W^Q *is* $Q-$*Brownian motion on space* $(\Omega, \mathcal{F}, Q, \mathcal{F}_t)$ *and,*

$$W_t^Q = W_t^P - \int_0^t \lambda_s ds \ Or \ dW^Q = dW^P - \lambda_t dt. \tag{2.7}$$

Therefore, in new S.D.E:

$$dX^Q = \mu(X^Q, t)dt + \sigma(X^Q, t)dW^Q, W_t^Q = W_t^P - \int_0^t \lambda_s ds,$$

on the other hand,

$$dX^p = (\mu(X^P, t) + \sigma(X^P, t)\lambda(t))dt + \sigma(x^P, t)dW^Q, \tag{2.8}$$

it means that by change of measure with exponential martingale as Radon Nikodem density, it just occurs change in drift and volatility remains such as before.

Theorem 2.4. *Suppose* $dX^P = \mu(X,t)dt + \sigma(X,t)dW^P$*, that* W^P *is* $P-$*Brownian motion and let*

$$\lambda(t) := -\mu(X,t)/\sigma(X,t).$$

Therefore, stochastic process X *under new measure* Q *such that* $\frac{dQ}{dP} = Z_t^\lambda$ *is* Q*-martingale and thus,* XZ_t^λ *is* P*-martingale.*

So $\lambda(t) = \frac{-\mu(X,t)}{\sigma(X,t)}$ is sufficient condition to be equivalence two following S.D.E.:

$$dX = \mu(X,t)dt + \sigma(X,t)dW \Leftrightarrow d(XZ_t^\lambda) = Z_t^\lambda(X\lambda(t) + \sigma(X,t))dW^P. \tag{2.9}$$

So, we could get the stochastic integral equation correspond to X (on account of, $Z_0^\lambda = 1$),

$$XZ_t^\lambda = \int_0^t (X\lambda(s) + \sigma(X,t))dW + X_0, \tag{2.10}$$

$$X = (Z_t^\lambda)^{-1}(\int_0^t Z_s^\lambda(X\lambda(s) + \sigma(X,t))dW + X_0).$$

By (2.10) and expectation from two side we have:

$$E^P[XZ_t^\lambda] = X_0 \Rightarrow E^P[X] = X_0(Z_t^\lambda)^{-1}. \tag{2.11}$$

$$E^P[(XZ_t^\lambda)^2] = X_0^2 + E\Big[\int_0^t (Z_s^\lambda)^2(X\lambda(s) + \sigma(X,t))^2 ds\Big] \quad \text{(by } \textit{itô} \text{ isometry)}$$

$$= X_0^2 + \int_0^t (Z_s^\lambda)^2 E\big[(X\lambda(s) + \sigma(X,t))^2\big]\big)ds$$

$$\mathrm{var}(XZ_t^\lambda) = \int_0^t (Z_s^\lambda)^2 E\big[(X\lambda(s) + \sigma(X,t))^2\big]\big)ds \tag{2.12}$$

Example 2.5. *Consider the following S.D.E.*

$$\begin{cases} dX = (a(t)\sqrt{X})dt + (b(t)\sqrt{X})dW, \\ X(0) = X_0. \end{cases} \tag{2.13}$$

from (2.11)*, we can get immediately* $E[X] = X_0(Z_t^\lambda)^{-1}$ *such that,* $\lambda = \frac{a(t)}{b(t)}$.

With numerical approximation by E.M. method we have:

$$\Delta X_i Z_{t_i}^\lambda = Z_{t_i}^\lambda (X_i \lambda(t_i) + \sigma_i)\Delta W_i$$
$$X_{t_{i+1}} Z_{t_{i+1}}^\lambda = X_{t_i} Z_{t_i}^\lambda + Z_{t_i}^\lambda (X_{t_i}\lambda(t_i) + \sigma_i)\Delta W_i$$
$$X_{t_{i+1}} = (Z_{t_{i+1}}^\lambda)^{-1} Z_{t_i}^\lambda (X_{t_i} + (X_{t_i}\lambda(t_i) + \sigma_i)\Delta W_i)$$

3. Converting S.D.E. to O.D.E As A Method

In this section, we consider the general S.D.E. as follows:

$$dX = f(X,t)dt + g(X,t)dW. \tag{3.1}$$

We decide to find an appropriate exponential process Z which XZ is the answer of an O.D.E. Thus, the two following equality are the same:

$$\underbrace{dX = f(X,t)dt + X\lambda(t)dW}_{(S.D.E.)} \Leftrightarrow \underbrace{d(XZ_t^{-\lambda(t)}) = Z_t^{-\lambda(t)}\big(f - \lambda(t)^2 X\big)dt}_{(O.D.E.)} \tag{3.2}$$

Example 3.1. *Consider the following S.D.E.*

$$\begin{cases} dX = (W^2 + 1)Xdt - WXdW, \\ X(1) = X_0. \end{cases}$$

again, with attention to initial condition of relation (3.2)*, we could have:*

$$dX = fdt + gdW \Leftrightarrow \begin{cases} dZ = -\frac{g}{X} Z dW & (1) \\ d(XZ) = Z\big(f - \frac{g^2}{X}\big)dt & (2) \end{cases}$$
$$dZ = W(t)ZdW \Rightarrow Z = Z_t^{W(t)}$$

Correspondent O.D.E. is $d(XZ_t^W) = Z_t^W X dt$ *that with change of variable* $v = XZ_t^W$ *and initial condition* $v(0) = X_0$*, we will get* $X = X_0(Z_t^W)^{-1}e^t$.

Another suitable case in converting S.D.E. to O.D.E. has been brought in $\emptyset$*ksendal*[5] (chapter 5, exercise 17).

Theorem 3.2. *The nonlinear stochastic differential equation*

$$dX = f(X,t)dt + c(t)XdW, \tag{3.3}$$

that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $C : \mathbb{R} \to \mathbb{R}$ *are given continuous functions can be written as:*

$$d(X(Z_t^c(t))^{-1}) = (Z_t^c(t))^{-1} f(X,t)dt, \tag{3.4}$$

that $Z_t^c(t)$ *is exponential Martingale process.*

Lemma 3.3. *From equality* (3.4)*, we have the following integral equation:*

$$X(Z_t^c(t))^{-1} = X_0 + \int_0^t (Z_s^c(t))^{-1} f(X,s)ds.$$

So as a numerical approximation we can get the following equality by E.M. method

$$\begin{aligned} X_{i+1}(Z_{t_{i+1}}^c(t))^{-1} &= X_i(Z_{t_i}^c(t))^{-1} + (Z_{t_i}^c(t))^{-1} f(X_i,t_i)\Delta t_i \\ X_{i+1} &= (Z_{t_i}^c(t))^{-1}(Z_{t_{i+1}}^c(t))(X_i + f(X_i,t_i)\Delta t_i) \\ X_{i+1} &= \exp((c(t_i)\Delta W_i - \frac{1}{2}c^2(t_i)\Delta t_i)(f(X_i,t_i)\Delta t_i + X_i) \end{aligned}$$

Also from equality (3.4)*, we have the following O.D.E.:*

$$d(X(Z_t^{c(t)})^{-1}) = (Z_t^{c(t)})^{-1} f((X(Z_t^{c(t)})^{-1})Z_t^{c(t)})dt$$

that with change of variable $V = X(Z_t^{c(t)})^{-1}$*, it is enough to solve*

$$\begin{cases} V' = (Z_t^{c(t)})^{-1} f(VZ_t^{c(t)}), \\ V(0) = X_0. \end{cases} \tag{3.5}$$

if even we want to solve this equation by numerical methods like Rung-Kutta, it has the order of convergence $\lambda = 3$ *such that numerical scheme like E.M. and Milstein have order of strong convergence* $\lambda = 0.5$ *and* $\lambda = 1$*, respectively* [7].

4. Change Of Variable Method

In this section, we intend to analyze change of variable method and find appropriate variable $u(Y) = X$ such that Y is the answer of well-known S.D.E.s to get explicitly solution of another arbitrary S.D.E.

$$dX = F(X,t)dt + G(X,t)dW, \quad X(0) = X_0.$$

We consider the following some various cases:
Case 1:

$$dY = a(t)dt + b(t)dW \tag{4.1}$$

By applying *itô* formula for $u(Y) = X$, on (4.1) we get:

$$\begin{cases} u'(a(t)) + \frac{1}{2}u''b^2(t) = F(u(Y), t) \\ u'b(t) = G(u(Y), t) \end{cases} \tag{4.2}$$

thus, it could be concluded that $\frac{a(t)}{b(t)}G + \frac{1}{2}GG' = F \Rightarrow \frac{F}{G} - \frac{1}{2}G' = \frac{a(t)}{b(t)}$.
So the relation $\frac{\partial}{\partial Y}\big(\frac{F}{G} - \frac{1}{2}G'\big) = 0$ is necessary condition to be solved an equation by change of variable on (4.1). $\Big(G' = \frac{\partial G}{\partial X}\Big)$

Example 4.1. *Consider the following stochastic model*

$$\begin{cases} dX = \frac{3}{4}t^2X^2dt + tX^{3/2}dW, \\ X(0) = 0. \end{cases}$$

It could be checked that for this equation the necessary condition holds. According to (4.2) $u'b(t) = tu^{3/2}$*. Since* u *is just a function of* Y*, so it should be* $b(t) = t$*,* $u = \frac{4}{Y^2}$ *and also* $\frac{a(t)}{b(t)} = 0$ *or* $a(t) = 0$*. Thus* $dY = tdW$ *and* $Y = \int_0^t sdW_s + Y(0)$*. Finally* $X = u(Y) = 4\big(\int_0^t sdW_s + Y(0)\big)^{-2}$.

Case 2:

$$\begin{cases} dY = f(Y, t)dt + c(t)YdW, \\ Y(0) = Y_0. \end{cases} \tag{4.3}$$

Example 4.2. *Consider the stochastic model such as follows:*

$$\begin{cases} dX = \frac{1}{2}\tan^3 dt + \tan XdW, \\ X(0) = \frac{\pi}{6}. \end{cases}$$

according to (4.2)*,*$Yu' = \tan(u)$*, and so,* $\sin(u) = Y$ *,* $X = u(Y) = \sin^{-1}(Y)$*. by* (4.3)*, we have*

$$f = \cos(u)\Big(\frac{1}{2}\tan^3(u) - \frac{1}{2}\frac{\tan^3(u)}{\sin^2(u)}\Big) = -\frac{1}{2}\sin(u).$$

Thus the exact solution of corresponding S.D.E. is $X = \sin^{-1}(Y)$ *that*

$$\begin{cases} dY = -\frac{1}{2}Ydt + YdW \\ Y(0) = \frac{1}{2} \end{cases}$$

This is a $Black - Scholes$ *equation so,* $X = \sin^{-1}\big(\frac{1}{2}\exp(W(t) - t)\big)$.

Case 3:

$$\begin{cases} dY = \beta(t)Ydt + \sigma(t)dW, \\ Y(0) = Y_0. \end{cases} \tag{4.4}$$

This equation is named *Langevin process* with exact solution.

$$Y = \exp(\int_0^s \beta(\theta)d\theta)\Big(Y(0) + \int_0^t \sigma(s)\exp\Big(\int_0^s -\beta(\theta)d\theta\Big)ds\Big).$$

By applying *itô* formula for $u(Y) = X$, on (4.4) we should have:

$$\begin{cases} u'\beta(t)Y + \frac{1}{2}u''\sigma^2(t) = F(u,t) \\ u'\sigma(t) = G(u,t) \end{cases}$$

if $G(u,t)$ could be separated as product of two different parts $G(u,t) = \sigma(t)\hat{G}(u)$, we have $u' = \hat{G}(u)$, $\hat{G}\beta(t)Y + \frac{1}{2}\hat{G}'\hat{G}\sigma^2(t) = F(u,t)$ and so $\beta(t) = \Big(F(u,t) - \frac{1}{2}\hat{G}'\hat{G}.\sigma^2(t)\Big)\frac{1}{Y.\hat{G}}$.

Therefore, the necessary condition to solve an S.D.E. by this method is so

$$\frac{\partial}{\partial u}\Big(\frac{1}{Y.\hat{G}}\Big(F - \frac{1}{2}\hat{G}\hat{G}'\sigma^2(t)\Big)\Big) = 0$$

Example 4.3. *Consider the following S.D.E. model*

$$\begin{cases} dX = (\sigma^2(t) + 2\beta(t)X)dt + 2\sigma(t)\sqrt{X}dW \\ X(0) = 0 \end{cases}$$

this equation is named "Squared Radial Langevin Process" that is a special case of C.I.R. investment rate model.

Since necessary condition in this example holds, so If $X = u(Y)$ we have $u(Y) = Y^2$, that Y is the solution of following langevin equation

$$\begin{cases} dY = \beta(t)Ydt + \sigma(t)dW \\ Y(0) = 0 \end{cases}$$

Therefore we can obtain $X = Y^2 = \Big(\int_0^t \sigma(s)\exp\big(\int_0^s -\beta(\theta)d\theta\big)ds\Big)^2$.

Case 4:

$$\begin{cases} dY = \frac{1}{2}B(Y)B'(Y)dt + B(Y)dW, \\ Y(0) = Y_0. \end{cases} \tag{4.5}$$

We can verify by *itô* formula that the answer of this S.D.E. with attention to $dZ = dW$, $Z(0) = Z_0$ and continuous function $h(x) = \int_0^x \frac{ds}{B(s)}$ is $Y = h^{-1}(W(t) + Z_0)$.

Lemma 4.4. *In general case, if* $\begin{cases} dZ = F(Z,t)dt + G(Z,t)dW, \\ Z(0) = Z_0. \end{cases}$ *, it could be inferred by itô formula on $Y = h^{-1}(Z)$ that;*

$$dY = (h^{-1}(Z))'dZ + \frac{1}{2}(h^{-1}(Z))''(dZ)^2$$

$$= B(Y)dZ + \frac{1}{2}B(Y)B'(Y)(dZ)^2$$
$$= (B(Y)F(Z,t) + \frac{1}{2}B(Y)B'(Y)G^2(Z,t))dt + B(Y)G(Z,t)dW$$

thus, we have the following S.D.E. with answer Y *such that* $h(Y) = \int_0^Y \frac{1}{B(s)}ds$.

$$\begin{cases} dY = \Big(B(Y)F(h(Y,t),t) + \frac{1}{2}B(Y)B'(Y)G^2(h(Y),t)\Big)dt + (B(Y)G(h(Y,t),t))dW \\ Y = h^{-1}\Big(\int_0^t F(Z,s)ds + \int_0^t G(Z,s)dW_s + Z_0\Big) \end{cases} \tag{4.6}$$

Example 4.5. *Consider C.I.R. investment rate model as follows:*

$$\begin{cases} dY = (\sigma^2(t) + b(t)Y)dt + 2\sigma(t)\sqrt{Y}dW, \\ Y(0) = Y_0 \end{cases}$$

with attention to (4.6)*, we have:*

$$\begin{cases} B(Y)G(h(Y),t) = 2\sigma(t)\sqrt{Y} \\ F(h(Y),t)B(Y) + \frac{1}{2}B(Y)B'(Y)G^2(h(Y),t) = \sigma^2(t) + b(t)Y \end{cases}$$

from first equality, if we put $B(Y) = 2\sqrt{Y}$*, then we have:*
$h(Y) = \sqrt{Y}, G(Y,t) = \sigma(t)$*. second equality gives the following result:*

$$F(h(Y),t) = \frac{b(t)}{2}\sqrt{Y} = \frac{b(t)}{2}h(Y) \Rightarrow F(Y,t) = \frac{b(t)}{2}Y,$$

thus, $Y = h^{-1}(Z)$, $dZ = \frac{b(t)}{2}Zdt + \sigma(t)dW$ *and finally we have*

$$Y = Z^2 = \Big(e^{\int_0^t \frac{b(s)}{2}ds}\big(\sqrt{Y_0} + \int_0^t e^{-\int_0^s \frac{b(r)}{2}dr}\sigma(s)dW_s\big)\Big)^2.$$

Special case of this example is $\begin{cases} dX = dt + 2\sqrt{X}dW, \\ X(0) = 0. \end{cases}$ *such that*$(\sigma(1) = 1$, $b(t) = 0)$*. the exact solution is* $X = Z^2$ *and* $dZ = dW$*. therefore,* $X = W^2$ *(this is true as we know* $dW^2 = dt + 2WdW$*.)*

Now in general case, we want to find an exact solution of $Cox-Ingersoll-Ross$ investment rate model:

$$\begin{cases} dY = (a(t) + b(t)Y)dt + \sigma(t)\sqrt{Y}dW, \\ Y(0) = Y_0. \end{cases} \tag{1}$$

such that

$$Y = h^{-1}(Z) = h^{-1}\Big(\int_0^t F(Z,t)dt + \int_0^t \sigma(s)ZdW + Z_0\Big)\ ,\ G(Z,t) = \sigma(t)Z \tag{2}$$

Converting S.D.E. to O.D.E. from (3.4), we have:

$$d\Big(\big(Z_t^\sigma\big)^{-1}Z\Big) = \big(Z_t^\sigma\big)^{-1}\big(A(t)Z^2 + B(t)Z + C(t)\big)dt.$$

Substituting $v = \big(Z_t^\sigma\big)^{-1}Z$, it is concluded:

$$\begin{aligned} v' &= A(t)Z_t^\sigma v^2 + B(t)v + (Z_t^\sigma)^{-1}c(t) \\ &= \hat{A}(t)v^2 + \hat{B}(t)v + \hat{C}(t). \end{aligned}$$

If $a(t) = \frac{\sigma^2(t)}{4}$, this equation is second-order Bernoli equation and it gets the previous examole. Also if $b(t) = 0$, The equation converts to a linear equation. Otherwise, this is a Riccati equation and it could be solved numerically for instance by *Runge-Kutta* method.

Example 4.6. *Consider C.I.R. model as follows:*

$$\begin{cases} dY = \sigma(t)Y^{3/2}dW, \\ Y(0) = Y_0. \end{cases}$$

By (4.6)*, we should have:*

$$\begin{cases} B(Y)F(h(Y),t) + \frac{1}{2}B(Y)B'(Y)\sigma^2(t)h^2(Y) = 0,\ (1) \\ B(Y)\sigma(t)h(Y) = \sigma(Y)Y^{3/2} \Rightarrow B(Y)h(Y) = Y^{3/2},\ (2) \\ B'(Y)h(Y) = \frac{3}{2}\sqrt{Y} - 1.\ (since : h(Y) = \int_0^Y \frac{ds}{B(s)}.)\ (3) \end{cases}$$

substituting (2) and (3) in (1), we have

$$F(h(Y),t) = \frac{\sigma^2(t)}{2}h(Y)\big(1 - \frac{3}{2}\sqrt{Y}\big) \Rightarrow F(Z,t) = \frac{\sigma^2(t)}{2}Z\big(1 - \frac{3}{2}\sqrt{h^{-1}(Z)}\big).$$

Assuming $h^{-1}(Z) = Z^2$, we get the following S.D.E. such that the implicit solution is, $Y = Z^2$.

$$\begin{cases} dZ = \Big(\frac{\sigma^2(t)}{2}Z - \frac{3}{4}\sigma^2(t)Z^2\Big)dt + \sigma(t)ZdW, \\ Z(0) = \sqrt{Y_0} \end{cases}$$

Converting to O.D.E., we reach to a second order Bernoulli equation ($v = Z(Z_t^\sigma)^{-1}$) :

$$d(Z(Z_t^\sigma)^{-1}) = (Z_t^\sigma)^{-1}\Big(\frac{\sigma^2(t)}{2}Z - \frac{3}{4}\sigma^2(t)Z^2\Big)dt$$

$$v' = \frac{\sigma^2(t)}{2}v - \frac{3}{4}\sigma^2(t)v^2$$

Afterwards, by change of variable $v^{-2} = u$, we have the linear equation

$$u' + \frac{\sigma^2(t)}{2}u = \frac{3}{4}\sigma^2(t)Z_t^\sigma$$

Finally, we have $Y = \left(\frac{1}{u}Z_t^\sigma\right)^2$ such that,

$$u = e^{-\int_0^t \frac{\sigma^2(s)}{2}ds}\left(\int_0^t e^{\int_0^s \frac{\sigma^2(r)}{2}dr}\frac{3}{4}\sigma^2(s)Z_s^\sigma ds + \frac{1}{Z_0^2}\right)$$

5. Conclusion

In this paper, we indicated that it could be solved linear S.D.E. or even linear O.D.E. from different orders by converting them to linear system equation and producing some independent noise in various coefficient. By this method we could also solve the linear differential equations even with nonhomogeneous part by stochastic differentials system.

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