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An approximate method to option pricing in the Heston model

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Abstract
The Heston model is one of the most popular stochastic volatility models for derivatives pricing, and it is a mathematical model describing the evolution of the volatility of an underlying asset. The model proposed by Heston(1993), takes into account non-lognormal distribution of the assets returns, leverage effect and the important mean-reverting property of volatility. In addition, it has a semi-closed form solution for European options. In this paper by means of classical Itô calculus, we decompose option prices as the sum of the classical Black-Scholes formula.This decomposition allows us to develop first- and second-order approximation formulas for option prices and implied volatilities in the Heston volatility framework, as well as to study their accuracy for short maturities. Moreover, we show that the corresponding approximations for the implied volatility are linear(first-order approximation) and quadratic(second-order approximation) in the log stock price.

Keywords and phrases: Stochastic volatility , Heston model, Itô calculus, Black-Scholes formula.

1. Introduction

Stochastic volatility models are a natural extension of the classical BlackScholes model that have been introduced as a way to manage the skew and smiles observed in real market data (see, for example, Hull and White [14], Scott [16], Stein and Stein [15], Ball and Roma [5] and Heston [13]). The study of these models has introduced new important mathematical and practical challenges, in particular related with the option pricing problem and the calibration of the corresponding parameters. In fact, we do not have closed-form option pricing formulas for the majority of the stochastic volatility models, and even in the case when closed-form pricing solutions can be derived (see, for example, Heston [13] or Schbel and Zhu [17]), they do not allow in general for fast calibration of the parameters. A recent trend in the literature has been the development of approximate closedform option pricing formulas. To this end, some authors have presented a perturbation analysis of the corresponding PDE with respect to a specific model parameter, like the volatility (see Hagan et al. [12]), the mean reversion (see Fouque et al. [10] and Fouque et al. [11]) or the

correlation (see Antonelli and Scarlatti [4]). In all these techniques, the region of validity of the results is restricted to either short or long maturities. The obtained approximations for option prices allow for fast calibration and give a better understanding of the role of model parameters. More recently, another approach has been proposed by Benhamou et al. [68], where the authors focus directly on the law of the log stock price at maturity, given its initial condition. They expand prices with respect to the volatility of the volatility, computing the correction terms using Malliavin calculus. This approach allows the authors to deal with short and long maturities, as well as with time-dependent coefficients. Another point of view has been presented in Als [1], where by means of Malliavin calculus the author extends the classical Hull and White formula by decomposing option prices as the sum of the same derivative price if there were no correlation and a correction due to correlation. As an application, the author develops a method to construct first-order option pricing approximation formulas that only require some regularity conditions (in the Malliavin calculus sense) for the volatility process and that can be applied for a very general class of volatility models, including the case of long-memory volatilities.
This paper is devoted to obtaining a new decomposition formula for option prices, similar to the one presented in Als [1], but valid even when the Malliavin regularity conditions needed in that work are not satisfied. Instead of expanding option prices around the Hull and White term by means of anticipating stochastic calculus (Malliavin calculus), we use the classical It formula to expand prices around the classical Black-Scholes formula with volatility parameter equal to the root-mean-square future average volatility. This will allow us to describe option prices as the sum of this last term plus a term due to the correlation and a term due to the volatility of the volatility. The paper is organized as follows. In Sect. 2, we introduce the main notations and hypotheses and prove our decomposition formula for option prices. In Sect. 3, we use the results from Sect. 2 to obtain first- and second-order option pricing approximation formulas.

2. A decomposition formula for option prices

We consider the Heston model for stock prices in a time interval [0,T] under a risk neutral probability P,

$$dS_t = rs_t dt + \sigma_t S_t(\rho dW_t + \sqrt{1-\rho^2}dB_t), t \in [0,T] \tag{1}$$

where

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \nu\sqrt{\sigma_t^2}dW_t$$

where r is the instantaneous interest rate (supposed to be constant),W and B are independent standard Brownian motions defined on a probability space (Ω, F, P),
$\rho \in [-1,1] and$ κ,θ and ν are constants satisfying the condition $2\kappa\theta > \nu^2$. It will be convenient in the following sections to make the change of variable $X_t = log(S_t)$, $t \in [0,T]$. It is well known that the price of a contingent claim of the form $h(X_t)$ at time t is given by

$$V_t = e^{-r(T-t)}E(h(X_t)|F_t),$$

where E denotes the expectation with respect to P.
We use the following notation:
$\upsilon_t^2 = \frac{1}{T-t}\int_t^T E(\sigma_s^2|F_t)ds,\ M_t = \int_0^t E(\sigma_s^2|F_t)ds.$
$BS(t,x,\sigma)$ will denote the price of a European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K and interest rate r. Remember that in this case
$BS(t,x,\sigma) = e^x N(d_2) - Ke^{-r(T-t)}N(d_1)$
where N denotes the cumulative probability function of the standard normal law and
$d_2 := \frac{x-x_t^*}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}, d_1 = \frac{x-x_t^*}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t}$
with $x_t^* := lnK - r(T-t)$
$L_{BS}(\sigma)$will denote the Black-Scholes differential operator (in the log variable) with volatilityσ,

$$L_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - r$$

lemma 2.1: Let $0 \leq t \leq s \leq T$. Then for every $n \geq 0$, there exists $c = c(n,\rho)$ such that

$$|\frac{\partial^n}{\partial x^n}(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})BS(s,x_s,\upsilon_s)| \leq C(\int_s^T E(\sigma_\theta^2|F_s)d\theta)^{-\frac{1}{2}(n+1)}$$

proof: [3]
Theorem 2.2:(Decomposition formula) Assume the model (1), where the volatility process $\sigma = \{\sigma_s, s \in [0,T]\}$ satisfies the condition $2\kappa\theta > \nu^2$. then for all $t \in [0,T]$,

$$\begin{aligned} V_t = BS(t,X_t;\upsilon_t) &+ \frac{\rho}{2}E(\int_t^T e^{-r(s-t)}H(s,X_s,\upsilon_s)\sigma_s d\langle M,W\rangle_s|F_t) \\ &+ \frac{1}{8}E(\int_t^T e^{-r(s-t)}K(s,X_s,\upsilon_s)d\langle M,M\rangle_s|F_t) \end{aligned} \quad (2)$$

where

$$H(s,X_s,\upsilon_s) := (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2})BS(s,X_s,\upsilon_s)$$

and

$$K(s, X_s, \upsilon_s) := (\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2})BS(s, X_s, \upsilon_s).$$

Proof. Notice that $BS(T, X_T, \upsilon_T) = V_T$. As $(e^{-rt}V_t)$ is a P -martingle, we can write

$$e^{-rt}V_t = E(e^{-rt}V_t|F_t) = E(e^{-rT}BS(T, X_T; \upsilon_T)|F_t)$$

Now our idea is to apply Itô's formula to the process $(e^{-rt}BS(t, X_t; \upsilon_t))$ .As the derivatives of $BS(t, x; y)$ are not bounded, we use an approximating argument.Take $\epsilon > 0$ and consider the process $(e^{-rt}BS(t, X_t; \upsilon_t^{\epsilon}))$,where $\upsilon_t^{\epsilon} := \sqrt{\frac{1}{T-t}(\epsilon + \int_t^T E(\sigma_s^2|F_t)ds)} = \sqrt{\frac{1}{T-t}(\epsilon + M_t - \int_0^t \sigma_s^2 ds)}$.
Applying the classical Itô formula and the relationship between the gamma, the Vega and the Delta

$$\frac{\partial BS}{\partial \sigma}(s, X_s, \upsilon_s^{\epsilon})\frac{1}{\upsilon_s^{\epsilon}(T-s)} = (\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})BS(s, X_s, \upsilon_s^{\epsilon}),$$ we deduce that

$$\begin{aligned} e^{-rT}BS(T, X_T; \upsilon_T^{\epsilon}) = e^{-rt}BS(t, X_t; \upsilon_t^{\epsilon}) &+ \int_t^T L_{BS}(\upsilon_s^{\epsilon})BS(s, X_s, \upsilon_s^{\epsilon})ds \\ &+ \int_t^T e^{-rs}(\frac{\partial BS}{\partial x})(s, X_s, \upsilon_s^{\epsilon})\sigma_s(\rho dW + \sqrt{1-\rho^2}dB) \\ &+ \frac{1}{2}\int_t^T e^{-rs}(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})BS(s, X_s, \upsilon_s^{\epsilon})dM_s \\ &+ \frac{\rho}{2}\int_t^T e^{-rs}(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2})BS(s, X_s, \upsilon_s^{\epsilon})\sigma_s d\langle M, W\rangle_s \\ &+ \frac{1}{8}\int_t^T e^{-rs}(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2})BS(s, X_s, \upsilon_s^{\epsilon})d\langle M, M\rangle_s \end{aligned}$$

Taking conditional expectations and multiplying by e^{rt}, we obtain that

$$\begin{aligned} e^{-r(T-t)}E(BS(T,X_T,v_T^{\epsilon})|F_t) &= BS(t,X_t,v_t^{\epsilon}) \\ &+ \frac{\rho}{2}E(\int_t^T e^{-r(s-t)}H(s,X_s,v_s^{\epsilon})\sigma_s d\langle M,W\rangle_s|F_t) \\ &+ \frac{1}{8}E(\int_t^T e^{-r(s-t)}K(s,X_s,v_s^{\epsilon})d\langle M,M\rangle_s|F_t). \end{aligned}$$

Letting now $\epsilon \searrow 0$, using the facts that $d\langle M,W\rangle_s = \nu\sigma_s \int_s^T e^{-\kappa(r-s)}drds$, $d\langle M,M\rangle_s = \nu^2\sigma_s^2(\int_s^T e^{-\kappa(r-s)}dr)^2ds$, Lemma 2.1 and the dominated convergence theorem, the result follows. □

Remark 2.3: Formula (2) gives us a tool to describe the impact on option prices of the correlation and the volatility of the volatility. Notice that the second term on the right-hand side of (2) becomes zero in the uncorrelated case $\rho = 0$.

3. Approximate option pricing formulas

This section presents a first- and a second-order approximation for option prices in the Heston volatility framework and their accuracy for short maturities.
lemma 3.1: Let $n > 1$ and $\delta := \frac{4\kappa\theta}{\nu^2} > n$ and $L(t) = (1 - e^{-kt})$ then
$E(\frac{1}{\sigma_s^n}|F_t) \leq C_n(T,\sigma_t)$
where $C_n(T,\sigma_t)$ is a positive constant which is nondecreasing as a function of T .

Proof. For the sake of simplicity, we take $t = 0$. From the proof of Lemma A.1 in Bossy and Diop [10], we can write

$$E(\frac{1}{\sigma_s^n}) \leq \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})L(s)^{\frac{n}{2}}} \int_0^1 u^{\frac{n}{2}-1}(1-u)^{\frac{2\kappa\theta}{\nu^2}-\frac{n}{2}-1} \times exp(-\frac{\sigma_0 e^{-\kappa s}u}{2L(s)})du.$$

Now, using the fact that $y^{\frac{n}{2}-1}e^{-y} \leq C(n)$ for some positive constant $C(n)$ and any $y > 0$, it follows that

$$E(\frac{1}{\sigma_s^n}) \leq \frac{C(n)}{L(s)}(\frac{e^{\kappa s}}{\sigma_0})^{\frac{n}{2}-1} \int_0^1 exp(-\frac{\sigma_0 e^{-\kappa s}u}{4L(s)})du \leq C(n)(\frac{e^{\kappa s}}{\sigma_0})^{\frac{n}{2}}$$

and this allows us to complete the proof.
Theorem 3.2 (First-order approximation formula) Assume the model (1), where the volatility process $\sigma = \{\sigma_s, s \in [0,T]\}$ satisfies the condition $2k\theta > \nu^2$. Then we have for all $t \in [0,T]$ that

$$
\begin{aligned}
&|V_t - BS(t, X_t; \upsilon_t) - \frac{\rho}{2} H(t, X_t, \upsilon_t) E(\int_t^T \sigma_s d\langle M, W\rangle_s | F_t)| \\
&\leq C(T, \sigma_t)\nu^2 (T-t)^{\frac{3}{2}}. \qquad (3)
\end{aligned}
$$

Proof: Consider the process $(e^{-rt} H(t, X_t; \upsilon_t) U_t)$, where

$$
U_t := \frac{\rho}{2} E(\int_t^T \sigma_s d\langle M, W\rangle_s | F_t)
$$

Since $U_T = 0$, the same arguments as in the proof of Theorem 2.2 allow us to write

$$
\begin{aligned}
0 &= H(t, X_t; \upsilon_t) U_t \\
&- \frac{\rho}{2} E(\int_t^T e^{-r(s-t)} H(s, X_s, \upsilon_s) \sigma_s d\langle M, W\rangle_s | F_t) \\
&+ \frac{\rho}{2} E(\int_t^T e^{-r(s-t)} (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}) H(s, X_s, \upsilon_s) U_s \sigma_s d\langle M, W\rangle_s | F_t) \\
&+ \frac{1}{8} E(\int_t^T e^{-rs} (\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}) H(s, X_s, \upsilon_s) U_s d\langle M, M\rangle_s | F_t)
\end{aligned}
$$

This, together with (2), gives us that

$$
\begin{aligned}
V_t &= BS(t, X_t; \upsilon_t) + H(t, X_t; \upsilon_t) U_t \\
&+ \frac{\rho}{2} E(\int_t^T e^{-r(s-t)} (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}) H(s, X_s, \upsilon_s) U_s \sigma_s d\langle M, W\rangle_s | F_t) \\
&+ \frac{1}{8} E(\int_t^T e^{-rs} (\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}) H(s, X_s, \upsilon_s) U_s d\langle M, M\rangle_s | F_t) \\
&+ \frac{1}{8} E(\int_t^T e^{-r(s-t)} K(s, X_s, \upsilon_s) d\langle M, M\rangle_s | F_t) \\
&= BS(t, X_t; \upsilon_t) + H(t, X_t; \upsilon_t) U_t + T_1 + T_2 + T_3. \qquad (4)
\end{aligned}
$$

Notice that

$$|U_s| \leq \frac{\nu\rho}{2} E(\int_s^T \sigma_r^2 \int_r^T e^{-k(u-r)} du dr | F_s)$$
$$= \frac{\nu\rho}{2} \int_s^T E(\sigma_r^2 | F_s) \int_r^T e^{-k(u-r)} du dr.$$

Then Lemma 2.1 gives us that

$$T_1 \leq C \frac{\nu^2 \rho^2}{4} E(\int_t^T e^{-r(s-t)} (\int_s^T E(\sigma_\theta^2 | F_s) d\theta)^{-\frac{5}{2}}$$
$$\times (\int_s^T E(\sigma_r^2 | F_s) \int_r^T e^{-k(u-r)} du dr) \sigma_s^2 \int_s^T e^{-k(u-s)} du ds | F_t)$$

Taking into account that $\int_s^T E(\sigma_\theta^2 | F_s) d\theta \geq \sigma_s^2 \int_s^T e^{-k(r-s)} dr$, it follows that

$$T_1 \leq C \frac{\nu^2 \rho^2}{4} \int_t^T e^{-r(s-t)} \sqrt{E(\sigma_s^{-2} | F_t)} (\int_s^T e^{-k(u-s)} du)^{\frac{1}{2}} ds.$$

Now Lemma 3.1 gives us that

$$T_1 \leq C(T, \sigma_t) \nu^2 \rho^2 \int_t^T e^{-r(s-t)} (\int_s^T e^{-k(u-s)} du)^{\frac{1}{2}} ds$$
$$\leq C(T, \sigma_t) \nu^2 \rho^2 (T-t)^{\frac{3}{2}}.$$

the same arguments give us that

$$T_2 \leq C(T, \sigma_t) \nu^2 \rho^2 (T-t)^2 \leq C(T, \sigma_t) \nu^2 \rho^2 (T-t)^{\frac{3}{2}}.$$

and

$$T_3 \leq C(T, \sigma_t) \nu^2 (T-t)^{\frac{3}{2}}.$$

Theorem 3.3 (Second-order approximation formula) Assume the model (1), where the volatility process $\sigma = \{\sigma_s, s \in [0, T]\}$ satisfies the condition

$2k\theta > \nu^2$. Then, we have for all $t \in [0,T]$ that

$$\begin{aligned}
&|V_t - BS(t, X_t; \upsilon_t) - \frac{\rho}{2} H(t, X_t, \upsilon_t) E(\int_t^T \sigma_s d\langle M, W\rangle_s | F_t) \\
&- \frac{1}{8} K(t, X_t, \upsilon_t) E(\int_t^T \sigma_s d\langle M, M\rangle_s | F_t)| \\
&\leq C(T, \sigma_t)(\nu^2 \rho^2 (T-t)^{\frac{3}{2}} + \nu^3 \rho (T-t)^2 + \nu^4 (T-t)^{\frac{5}{2}})
\end{aligned}$$

Proof: For the process $(e^{-rt} K(t, X_t; \upsilon_t) R_t)$ with $R_t := \frac{1}{8} E(\int_t^T d\langle M, M\rangle_s | F_t)$, we have $R_T = 0$ and so the same arguments as in the proof of Theorem 3.2 give us that

$$\begin{aligned}
0 &= K(t, X_t; \upsilon_t) R_t \\
&- \frac{1}{8} E(\int_t^T e^{-r(s-t)} K(s, X_s, \upsilon_s) d\langle M, M\rangle_s | F_t) \\
&+ \frac{\rho}{2} E(\int_t^T e^{-r(s-t)} (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}) K(s, X_s, \upsilon_s) R_s \sigma_s d\langle M, W\rangle_s | F_t) \\
&+ \frac{1}{8} E(\int_t^T e^{-r(s-t)} (\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}) K(s, X_s, \upsilon_s) R_s d\langle M, M\rangle_s | F_t)
\end{aligned}$$

This, together with (2) and (3), allows us to write

$$\begin{aligned}
V_t &= BS(t, X_t; \upsilon_t) + H(t, X_t; \upsilon_t) U_t + K(t, X_t; \upsilon_t) R_t \\
&+ \frac{\rho}{2} E(\int_t^T e^{-r(s-t)} (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}) H(s, X_s, \upsilon_s) U_s \sigma_s d\langle M, W\rangle_s | F_t) \\
&+ \frac{1}{8} E(\int_t^T e^{-rs} (\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}) H(s, X_s, \upsilon_s) U_s d\langle M, M\rangle_s | F_t) \\
&+ \frac{\rho}{2} E(\int_t^T e^{-r(s-t)} (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}) K(s, X_s, \upsilon_s) R_s \sigma_s d\langle M, W\rangle_s | F_t) \\
&+ \frac{1}{8} E(\int_t^T e^{-rs} (\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}) K(s, X_s, \upsilon_s) R_s d\langle M, M\rangle_s | F_t) \\
&= BS(t, X_t; \upsilon_t) + H(t, X_t; \upsilon_t) U_t + K(t, X_t; \upsilon_t) R_t + T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

Then, by the proof of Theorem 3.2

$$T_1 + T_2 \leq C(T, \sigma_t) \nu^2 \rho^2 (T-t)^{\frac{3}{2}}.$$

On the other hand,

$$
\begin{aligned}
|R_s| &\leq \frac{\nu^2}{2} E\left(\int_s^T \sigma_r^2 \left(\int_r^T e^{-k(u-r)} du\right)^2 dr | F_s\right) \\
&= \frac{\nu^2}{2} \int_s^T E(\sigma_r^2 | F_s) \left(\int_r^T e^{-k(u-r)} du\right)^2 dr.
\end{aligned}
$$

Then, using Lemma 2.1, we obtain as in the proof of Theorem 2.2

$$
\begin{aligned}
T_3 &\leq \frac{\nu^3 \rho}{2} E\Big(\int_t^T e^{-r(s-t)} \Big(\int_s^T E(\sigma_\theta^2 | F_s) d\theta\Big)^{-3} \\
&\times \Big(\int_s^T E(\sigma_r^2 | F_s) \Big(\int_r^T e^{-k(u-r)} du\Big)^2 dr\Big) \\
&\times \sigma_s^2 \Big(\int_s^T e^{-k(u-s)} du\Big) ds | F_t\Big) \\
&\leq C(t, \sigma_t) \nu^3 \rho \int_t^T e^{-r(s-t)} E(\sigma_s^{-2} | F_t) \int_s^T e^{-k(r-s)} dr ds.
\end{aligned}
$$

Now, by Lemma 3.1,

$$T_3 \leq C(T, \sigma_t) \nu^3 \rho (T-t)^2.$$

the same arguments give us that

$$T_4 \leq C(T, \sigma_t) \nu^4 (T-t)^{\frac{5}{2}}.$$

Remark3.4: As in Theorem 3.2, the accuracy of this second-order approximation increases when the volatility of the volatility or the time to maturity decreases. On the other hand, we can observe that when $\rho = \pm 1$, the accuracy of Theorem 3.3 is of the same order $O(\nu^2)$ as in Theorem 3.2, while when the correlation decreases, this accuracy becomes significantly better and is of the order $O(\nu^4)$ when $\rho = 0$

4. Conclusion

By means of classical Itô calculus, we have decomposed option prices in the Heston volatility framework as the sum of the classical Black-Scholes formula, with volatility parameter equal to the root-mean-square future average volatility, plus a term due to correlation and a term due to the volatility of the volatility. This decomposition formula allows us to construct first- and second-order option pricing approximation formulas that are extremely easy to compute, as well as to study their accuracy for short maturities. Moreover, we have seen that the corresponding approximations for the implied volatility are linear (first-order approximation) and quadratic (secondorder approximation) in the log stock price variable x. The presented methods need only some general integrability conditions and extend some recent results in Als and Ewald [2].

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