An Efficient Numerical Approximation of the American Option Pricing Problem

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Abstract:

This paper deals with developing an efficient numerical approximation of the American option pricing problem as a free boundary problem. The recently introduced artificial boundary conditions of Han and Wu [٩] are also employed. In order to solve the problem, a finite difference method is applied. The research has also taken advantage of the numerical approximation of the free boundary near expiry. Comparing the results coming from this method with those of the former methods, this research has been able to increase the accuracy of the commonly used methods.

Keywords: American option, Artificial boundary condition, Finite difference method, Free boundary approximation, Partial differential equations

١. Introduction

American-style options enable the holder to exercise the option at any point in time up to the maturity. The pricing and hedging of American options have been considered as a long-standing problem in computational finance calling for effective models for option pricing such as the Black-Scholes equation [٢]. Due to the complex nature of the problem and the prominent rule played by it in current financial markets, the last twenty years have witnessed an intense body of research activities to resolve the problem.

Although some analytical expressions for the price process have been obtained in the literature, there is no universal, explicit and easily computable formula currently available for the general case. This has resulted in the introduction of different analytical and semi-analytical tools to solve the problem approximately based on the Black-Scholes equation, with which the price of these options is described [٨, ١١, ١٤, ٢١]. However, these methods are along with some drawbacks which can be attributed to the fact that they could not easily be extended beyond the Black-Scholes framework.

With regard to this problem, researchers have devised and analyzed a variety of numerical approaches such as the binomial method, finite difference method, finite element method and spectral method, with which the pricing problem can be solved efficiently [٦, ٣, ١, ٤]. Besides, these approaches could be implemented in conjunction with different formulations like front-tracking transformations, front-fixing transformations, penalty methods, operator splitting ideas and singularity separating techniques [٧, ١٠, ١٣, ٥].

Truncating the infinite domain and employing it in the context of front-tracking ideas employed in Han and Wu [٩], Pantazopoulos et al. [١٥], Tangman et al. [١٦] and Toivanen [١٨] has been known as one of the developed models. Han and Wu [٩] have considered the American option pricing problem with a free boundary based on some properties of the solution of the Black–Scholes equation and an artificial Neumann boundary condition used at the other end of the domain. By an appropriate change of variables, the Black-Scholes PDE with free boundary can be transformed to the well-known heat equation on an infinite domain. Regarding existence of the different types of analytic and numerical methods for the heat equation transformation, this is a suitable choice.

In dealing with problems defined on infinite or semi-infinite domains, the introduction of artificial boundary conditions has a special significance. In fact, the artificial boundary conditions include the type of methods transforming the infinite domain into a finite domain. These techniques use the Duhamel's theorem, Green's functions, Fourier transforms and Laplace transforms [٧, ٩, ١٧, ٢٠]. Some improvements on this basic idea have been reported by Tangman and his co-workers [١٦] based on a singularity separation framework in the context of compact finite difference schemes. In addition, Ehrhardt and Mickens [٧] have applied an approximate discrete transparent boundary condition with a kernel having the form of a finite sum-of-exponentials.

The main focus of this paper is to modify the Han and Wu's formulation. In order to achieve it, front tracking is applied and a complete treatment is given, in which the behavior of the free boundary and an artificial Neumann boundary condition used at the other end of the domain is taken into account. Solving the equation could be done based on an implicit or explicit time-stepping method where in [٩] the Crank-Nicolson method is adopted due to its unconditional stability. We seek high order approximations for the free-boundary. In order to compare the accuracy and efficiency of our numerical methods, we employ the binomial approximation with enough division of Nelson and Ramaswamy [١٢] and Han &Wu formulation [٩] as the benchmark approaches. In this paper the basic instrument to improve the accuracy of the original approach of Han and Wu is the appropriate changes. However their approach does not take into account the smooth pasting condition which affects the accuracy of the numerical solutions. But this condition with the numerical approximation of the free boundary leads to obtain the final value which will be useful for solving the heat equation numerically. These will provide us with an exact description of the early exercise boundary resulting in more accurate estimates for the price itself.

The remainder of the paper is organized as follows. Section ٢ presents the Black–Scholes equation and recalls the standard transformations to a forward-in-time heat equation. Section ٣ formulates the analytic artificial boundary approach for the heat equation. Section ٤ is described the numerical approximation of the problem. The numerical example is given in Section ٥. Section ٦ concludes.

٢. The Black–Scholes Equation

The ultimate aim is to find the put option price $P(S,t)$ in which S is the asset price and t is the elapsed time. Let r, σ, T denote respectively the risk-free interest rate, the volatility of the asset price and the expiration date. Then a partial differential equation for the value function $P(S,t)$ is given by the free boundary value problem of the Black–Scholes type of the form (cf. [١٩]);

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{١}{٢}\sigma^{٢} S^{٢} \dfrac{\partial^{٢} P}{\partial S^{٢}} + rS\dfrac{\partial P}{\partial S} - rP = ٠ & S_f(t) < S < +\infty, \quad ٠ \le t < T, \qquad (١) \\ P(S,T) = \max\{K - S, ٠\}, & K = S_f(T) \le S < +\infty, \qquad (٢) \\ P(S,t) = K - S, \quad \dfrac{\partial P}{\partial S}(S,t) = -١, & S \le S_f(t), \quad ٠ \le t \le T, \qquad (٣) \\ P(S,t) \to ٠ \quad as \quad S \to +\infty, & ٠ \le t \le T, \qquad (٤) \end{cases}$$

We introduce the change of variable:

$$t = T - \frac{٢\tau}{\sigma^{٢}}$$

Let $S = Ke^{x}$, $\tilde{r} = \frac{٢r}{\sigma^{٢}}$, $S_f(t) = Ke^{S(\tau)}$, and $P(S,t) = Kp(x,\tau)$.

Then the problem becomes, for the transformed price $p(x,\tau)$ and the optimal exercise boundary $x = S(\tau)$,

$$\begin{cases} \frac{\partial p}{\partial \tau} - \tilde{r}\frac{\partial p}{\partial x} - \frac{\partial^{٢} p}{\partial x^{٢}} + \frac{\partial p}{\partial x} + \tilde{r}p = ٠ & S(\tau) < x < +\infty, \quad ٠ < \tau < \frac{T\sigma^{٢}}{٢}, \quad (٥) \\ p(x,٠) = \max\{١ - e^{x}, ٠\} & S(٠) < x < +\infty, \quad ٠ < \tau < \frac{T\sigma^{٢}}{٢}, \quad (٦) \\ p(x,\tau) = ١ - e^{x}, \quad p_x(x,\tau) = -e^{x} & x \le S(\tau), \quad ٠ \le \tau \le \frac{T\sigma^{٢}}{٢}, \quad (٧) \\ p(x,\tau) \longrightarrow ٠ \quad as \quad x \longrightarrow +\infty & , \quad ٠ \le \tau \le \frac{T\sigma^{٢}}{٢}, \quad (٨) \end{cases}$$

Besides, in order to solve the set of equations (٥)-(٨) we also utilize a similar scheme as Han & Wu approach. Let

$$\alpha = -\frac{\tilde{r} - ١}{٢}, \qquad \beta = -\frac{(\tilde{r} + ١)^{٢}}{٤}$$

Furthermore, we introduce the change of variables

$$g(x) = max\{١ - e^{x}, ٠\}, \quad p(x,\tau) = e^{\alpha x + \beta \tau} q(x,\tau) \qquad (٩)$$

Then equations (٥) and (٦) are equivalent to the following equations:

$$\begin{cases} q_\tau = q_{xx}, & S(\tau) < x < +\infty, \quad ٠ < \tau < \frac{T\sigma^{٢}}{٢}, \quad (١٠) \\ q(x,٠) = e^{-\alpha x} g(x) & S(٠) < x < +\infty, \quad (١١) \end{cases}$$

In far-field approach, placing the boundary to the finite domain is useful. Thus, an artificial boundary can be utilized for these implementation problems.

٣. An exact boundary condition on the artificial boundary

In this section, we derive a class of exact artificial boundary conditions for an unbounded domain. The domain is truncated to a finite domain and the finite difference method is employed to the equation.

Unbounded domain: $\bar{\Omega} = \{(x,\tau) \mid S(\tau) < x < +\infty, \quad ٠ < \tau \le \frac{T\sigma^{٢}}{٢}\}$

Artificial boundary: $\Gamma_b = \{(x,\tau) \mid x = b, \quad ٠ < \tau \le \frac{T\sigma^{٢}}{٢}\}$

External field: $\Omega_e = \{(x,\tau) \mid b < x < +\infty, \quad ٠ < \tau \le \frac{T\sigma^{٢}}{٢}\}$

Internal field: $\Omega_i = \{(x,\tau) \mid S(\tau) < x < b, \quad ٠ < \tau \le \frac{T\sigma^{٢}}{٢}\}$

On Ω_e, the solution of the problem, $q(x,\tau)$, satisfies

$$q_\tau = q_{xx} \qquad b < x < +\infty, \quad ٠ < \tau \le \frac{T\sigma^٢}{٢} \qquad (١٢)$$

$$q(x,٠) = ٠ \qquad b < x < +\infty \qquad (١٣)$$

If we know the value of $q(x,\tau)$ on the boundary Γ_b,

$$q(b,\tau) = \varphi(\tau)$$

We could use the Duhamel's theorem for solutions of the PDE

$$q(x,\tau) = \frac{x-b}{٢\sqrt{\pi}} \int_٠^\tau e^{-\frac{(x-b)^٢}{٤(t-\lambda)}} \frac{\varphi(\lambda)}{(t-\lambda)^{٣/٢}} d\lambda \qquad (١٤)$$

$$\frac{\partial q(x,\tau)}{\partial x} = \frac{-١}{\sqrt{\pi}} \int_٠^\tau \frac{\partial q(b,\lambda)}{\partial \lambda} \frac{d\lambda}{(t-\lambda)^{١/٢}} \qquad (١٥)$$

So, the proposed problem is

$$\begin{cases} q_\tau = q_{xx} & S(\tau) < x < +\infty, \quad ٠ < \tau < \frac{T\sigma^٢}{٢}, \quad (١٦) \\ q(x,٠) = e^{-\alpha x} g(x) & S(٠) < x < +\infty, \quad (١٧) \\ \alpha e^{\alpha S(\tau)+\beta\tau} q(S(\tau),\tau) + e^{\alpha S(\tau)+\beta\tau} q_x(S(\tau),\tau) + e^{S(\tau)} = ٠ & ٠ \le \tau \le \frac{T\sigma^٢}{٢}, \quad (١٨) \\ \frac{\partial q(x,\tau)}{\partial x} = \frac{-١}{\sqrt{\pi}} \int_٠^\tau \frac{\partial q(b,\lambda)}{\partial \lambda} \frac{d\lambda}{(\tau-\lambda)^{١/٢}} & (١٩) \end{cases}$$

Although all finite difference methods for the boundary value problem are possible choices, we consider the Crank-Nicolson scheme with an evenly spaced grid, as it is the most widely used method for option pricing.

٤. Finite Difference Schemes

The systems are set up in the interval $[x_m, x_p]$, where $x_m < S(\tau)$ and $x_p = b > S(٠)$ for all $\tau > ٠$,

$$\tau_m = m\delta_\tau, \qquad m = ٠,١,٢,\ldots$$
$$x_n = b - n\delta_x, \qquad n = ٠,١,٢,\ldots$$

Denote the approximate solution of $q(x_n, \tau_m)$ by q_n^m.

By using the Crank-Nicolson finite difference,

$$\frac{q_n^m - q_n^{m-١}}{\delta_\tau} = \frac{١}{٢}\left(\frac{q_{n+١}^{m-١} - ٢\, q_n^{m-١} + q_{n-١}^{m-١}}{(\delta_x)^٢} + \frac{q_{n+١}^m - ٢\, q_n^m + q_{n-١}^m}{(\delta_x)^٢}\right)$$

Letting $\rho = \delta_\tau / (\delta_x)^٢$, we have

$$-\frac{\rho}{٢} q_{n-١}^m + (١+\rho)\, q_n^m - \frac{\rho}{٢} q_{n+١}^m = \frac{\rho}{٢} q_{n-١}^{m-١} + (١-\rho)\, q_n^{m-١} + \frac{\rho}{٢} q_{n+١}^{m-١} \qquad (٢٠)$$

And the initial value will be

$$q_n^٠ = \exp(-\alpha x_n)(١ - \exp(x_n))^+ . \qquad (٢١)$$

We provide the approximations to $s(\tau)$ for determining the location of the free boundary in the numerical schemes (cf. [٥]);

$$\begin{cases} \xi(\tau) = log\sqrt{٤\pi\tilde{r}^{٢}\tau}\,, & (٢٢) \\ \alpha(\tau) = -\xi(\tau) - \dfrac{١}{٢(\xi(\tau)-d)} + \dfrac{\frac{١}{٨}+\frac{d}{٨}}{(\xi(\tau)-d)^{٢}}, \quad d = ٠.٩٦٦٢١\ldots, & (٢٣) \\ s(\tau) = \sqrt{٤\tau\alpha(\tau)} & (٢٤) \end{cases}$$

An explicit finite difference and (١٨) lead to obtain the final value;

$$q_{N+١}^{m} = (١-\alpha\delta_x)\left(\rho q_{N+١}^{m-١} + (١-٢\rho)\,q_N^{m-١} + \rho\, q_{N-١}^{m-١}\right) - \delta_x e^{(١-\alpha)S(\tau)-\beta\tau} \qquad (٢٥)$$

From mean value theorem for integration, corresponding equation of the artificial boundary condition is approximated by

$$q_{n+١}^{m} = q_{n-١}^{m} - \frac{٤\delta_x}{\sqrt{\pi\delta_\tau}}\sum_{j=١}^{m}\frac{q_n^{j}-q_n^{j-١}}{\sqrt{m-j}+\sqrt{m-j+١}}$$

Besides, by applying a purely implicit scheme to the heat equation at $x_{٠} = b$ and above equation

$$q_{٠}^{m} = \frac{١}{١+٢\rho-\frac{٤\sqrt{\rho}}{\sqrt{\pi}}}\left(٢\rho q_{١}^{m} + \left(١-\frac{٤\sqrt{\rho}}{\sqrt{\pi}}\right)q_{٠}^{m-١} + \frac{٤\sqrt{\rho}}{\sqrt{\pi}}\sum_{j=١}^{m-١}\frac{q_{٠}^{j}-q_{٠}^{j-١}}{\sqrt{m-j}-\sqrt{m-j+١}}\right) \qquad (٢٦)$$

In a much more condensed way, Eq. (١٩) can be written in the following matrix form:

$$AQ_n^{m} = BQ_n^{m-١} + e \qquad (٢٧)$$

with

$$Q_n^{m} = (q_{١}^{m}, q_{٢}^{m}, \ldots, q_{N-١}^{m})^{T}$$
$$Q_n^{m-١} = (q_{١}^{m-١}, q_{٢}^{m-١}, \ldots, q_{N-١}^{m-١})^{T}$$
$$e = (f, ٠, \ldots, ٠, g)^{T}$$

and

$$A = \begin{bmatrix} a & -b & ٠ & ٠ & ٠ & & ٠ \\ -b & c & -b & ٠ & ٠ & & ٠ \\ ٠ & -b & c & -b & ٠ & & ٠ \\ ٠ & \ddots & \ddots & \ddots & & & \\ \vdots & & & & -b & c & -b \\ ٠ & \cdots & & & & -b & c \end{bmatrix} \qquad B = \begin{bmatrix} d & b & ٠ & ٠ & ٠ & & ٠ \\ b & d & b & ٠ & ٠ & & ٠ \\ ٠ & b & d & b & ٠ & & \\ ٠ & \ddots & \ddots & \ddots & & & ٠ \\ \vdots & & & ٠ & b & d & b \\ ٠ & \cdots & & & ٠ & b & d \end{bmatrix}.$$

The coefficients in the matrices are:

$$\begin{cases} a = ١ + \rho - \frac{\rho^٢}{H} \\ b = \frac{\rho}{٢} \\ c = ١ + \rho \\ d = ١ - \rho \\ f = \frac{\rho}{٢}\left(١ + \frac{١}{H} - \frac{٤\rho\sqrt{\rho}}{H\sqrt{\pi}}\right) q_{.}^{m-١} + \frac{٢\rho\sqrt{\rho}}{H\sqrt{\pi}} \sum_{j=١}^{m-١} \frac{q_{.}^{j} - q_{.}^{j-١}}{\sqrt{m-j} - \sqrt{m-j+١}} \\ g = \frac{\rho}{٢}\left(q_N^m + q_N^{m-١}\right) \end{cases}$$

Which $H = ١ + ٢\rho - \frac{٤\sqrt{\rho}}{\sqrt{\pi}}$.

We summarize the calculation of the price $P(S,t)$ for the American put option in the Algorithm given in Appendix A.

٥. Computational results

To compare the algorithm with the standard finite difference approximations, we computed an example of put options. The comparisons are based on the accuracy of the approximate option values. Since the exact option values are unknown, we use the binomial method to find the option values. All the algorithms are implemented using MATLAB for testing purposes. The linear systems are constructed using the Crank-Nicolson finite difference method in the interval $[a, b]$ in which the left boundary "a" moves until the free boundary and "b" is an artificial boundary.

Example ١. Consider a three-week American put option without dividend rate. The exercise price is $١٠٠, the risk-free interest rate is $r = ٠.٠٣$, and the volatility is ٤٠٪ per annum. The left boundary is set to $a = -١.٤$. (The minimum value of $S(\tau)$ is about -١.٢٩.) A step size $n = ٤٠$ with ratio $\rho = ٠.٤٩$ is taken for the method.

In order to control the unbounded domain, the boundary conditions are taken distant from the area of interest. Thus, even in case of facing unsuitable boundary conditions, the solution will be kept from being affected. Besides, in order to achieve a more accurate result, a large domain should be taken for granted. However, in such cases, the points should be controlled to achieve evenly spaced grids. It can be seen that the computational time rises in such cases.

Totally, in spite of a small domain is leading to fast computation, a large domain with a slower computational time is preferred. It is due to the fact that in case of the small domain, the accuracy can be affected by the error related to the imposed boundary conditions.

Figure ٥.١ shows the value $P(S,t)$ of an American put with b= ١.٨. The plot uses a strike price is $١٠٠, a time to maturity of three weeks, an interest rate of ٣٪, and the volatility of ٤٠٪. As shown in figure ٥.١, for some "m" which is large, the numbers of accurate solution are more and the plot is the same of normal one.

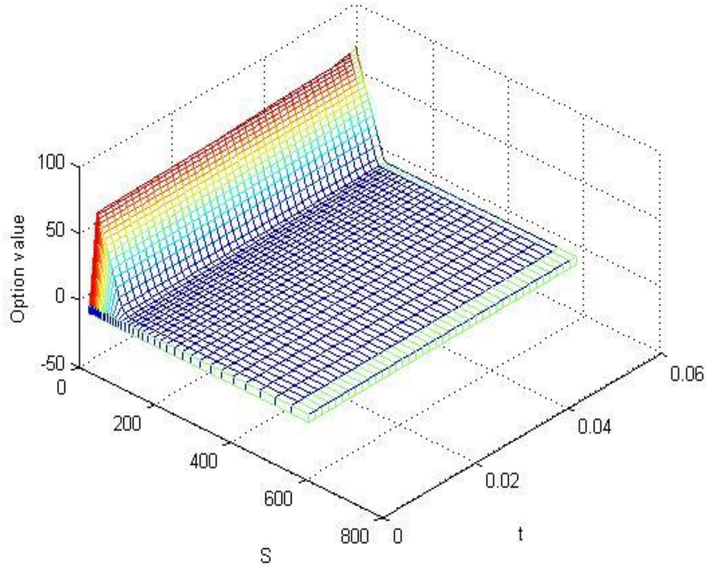

Fig. ٥.١. Value $P(S,t)$ of an American put with *r* = ٠.٠٣, *σ* = ٠.٤٠, *K* = ١٠٠, *T* = ٢١/٣٦٥, *b*= ١.٨

Table ١: Numerical results for Example ١

b	Asset Price	Proposed Method	Binomial Method
١.٨	٦٩.٧٦٧٦٣٢٢٦٠٧١٠٣١	٣٠.٢٣٢٣٦٧٣٩٢٨٩٦٨٧٩	٣٠.٢٤٣٧٨٣١٧٥٦٦٠١
	٦٤.٤٠٣٦٤٢١٠٨٣١٤١	٣٥.٥٩٦٣٥٧٨٩١٦٨٥٨٥٥	٣٥.٥٩٦٣٥٧٨٩١٦٨٥٩
	٥٩.٤٥٢٠٥٤٧٩٧٠١٩٤	٤٠.٥٤٧٩٤٥٢٠٢٩٨٠٥٦٤	٤٠.٥٤٧٩٤٥٢٠٢٩٨٠٦
	٥٤.٨٨١١٦٣٦٠٩٤٠٢٧	٤٥.١١٨٨٣٦٣٩٠٥٩٧٣٤٥	٤٥.١١٨٨٣٦٣٩٠٥٩٧٣
	٥٠.٦٦١٦٩٩٢٣٦٥٥٩٠	٤٩.٣٣٨٣٠٠٧٦٣٤٤١٠٣٣	٤٩.٣٣٨٣٠٠٧٦٣٤٤١٠
	٤٦.٧٦٦٦٤٢٧٠٠٩٩٠٩	٥٣.٢٣٣٣٥٧٢٩٩٠٠٩٠٧٢	٥٣.٢٣٣٣٥٧٢٩٩٠٠٩٠
	٤٣.١٧١٠٥٢٣٤٢٩٠٨٠	٥٦.٨٢٨٩٤٧٦٥٧٠٩٢٠٢٣	٥٦.٨٢٨٩٤٧٦٥٧٠٩٢٠
	٣٩.٨٥١٩٠٤١٠٨٤٥١٤	٦٠.١٤٨٠٩٥٨٩١٥٤٨٥٨٦	٦٠.١٤٨٠٩٥٨٩١٥٤٨٥
	٣٦.٧٨٧٩٤٤١١٧١٤٤٢	٦٣.٢١٢٠٥٥٨٨٢٨٥٥٧٧٢	٦٣.٢١٢٠٥٥٨٨٢٨٥٥٧
	٣٣.٩٥٩٥٥٢٥٦٤٤٩٣٩	٦٦.٠٤٠٤٤٧٤٣٥٥٠٦٠٦٧	٦٦.٠٤٠٤٤٧٤٣٥٥٠٦٠
	٣١.٣٤٨٦١٨٠٨٨٢٦٠٥	٦٨.٦٥١٣٨١٩١١٧٣٩٤٦٣	٦٨.٦٥١٣٨١٩١١٧٣٩٥
	٢٨.٩٣٨٤٢١٧٩٣٩٠٥١	٧١.٠٦١٥٧٨٢٠٦٠٩٤٩٣٦	٧١.٠٦١٥٧٨٢٠٦٠٩٤٩
	٢٦.٧١٣٥٣٠١٩٦٥٨٥٠	٧٣.٢٨٦٤٦٩٨٠٣٤١٤٩٦٠	٧٣.٢٨٦٤٦٩٨٠٣٤١٥٠
	٢٤.٦٥٩٦٩٦٣٩٤١٦٠٦	٧٥.٣٤٠٣٠٣٦٠٥٨٣٩٣٣٨	٧٥.٣٤٠٣٠٣٦٠٥٨٣٩٤
٢.٢	٧٢.٦١٤٩٠٣٧٠٧٣٦٩١	٢٧.٣٨٥٠٩٦٢٩٢٦٣٠٨٩١	٢٧.٤٧٣٩٦٤٤٧٠٥٩٩٩
	٦٦.٣٦٥٠٢٥٠١٣٦٣١٩	٣٣.٦٣٤٩٧٤٩٨٦٣٦٨٠٥٧	٣٣.٦٣٤٩٧٤٩٨٦٣٦٨١
	٦٠.٦٥٣٠٦٥٩٧١٢٦٣٣	٣٩.٣٤٦٩٣٤٠٢٨٧٣٦٦٦٢	٣٩.٣٤٦٩٣٤٠٢٨٧٣٦٧
	٥٥.٤٣٢٧٢٨٤٧٣٤٥٠٧	٤٤.٥٦٧٢٧١٥٢٦٥٤٩٢٩١	٤٤.٥٦٧٢٧١٥٢٦٥٤٩٣
	٥٠.٦٦١٦٩٩٢٣٦٥٥٩٠	٤٩.٣٣٨٣٠٠٧٦٣٤٤١٠٣٣	٤٩.٣٣٨٣٠٠٧٦٣٤٤١٠
	٤٦.٣٠١٣٠٦٨٣١١٢٢٨	٥٣.٦٩٨٦٩٣١٦٨٨٧٧١٨٥	٥٣.٦٩٨٦٩٣١٦٨٨٧٧٢
	٤٢.٣١٦٢٠٨٢٣١٧٧٤٩	٥٧.٦٨٣٧٩١٧٦٨٢٢٥١٠٦	٥٧.٦٨٣٧٩١٧٦٨٢٢٥١
	٣٨.٦٧٤١٠٢٣٤٥٤٥٠١	٦١.٣٢٥٨٩٧٦٥٤٥٤٩٨٧٧	٦١.٣٢٥٨٩٧٦٥٤٥٤٩٩
	٣٥.٣٤٥٤٦٨١٩٥٨٧٨٠	٦٤.٦٥٤٥٣١٨٠٤١٢٢٠٠١	٦٤.٦٥٤٥٣١٨٠٤١٢٢٠
	٣٢.٣٠٣٣٢٥٦٤٢٢٢٥٣	٦٧.٦٩٦٦٧٤٣٥٧٧٧٤٦٩٧	٦٧.٦٩٦٦٧٤٣٥٧٧٧٤٧
	٢٩.٥٢٣٠١٦٦٩٢٤٠١٤	٧٠.٤٧٦٩٨٣٣٠٧٥٩٨٥٧٣	٧٠.٤٧٦٩٨٣٣٠٧٥٩٨٦
	٢٦.٩٨٢٠٠٥٦٣٨٤٦٨٧	٧٣.٠١٧٩٩٤٣٦١٥٣١٣٠٨	٧٣.٠١٧٩٩٤٣٦١٥٣١٣
	٢٤.٦٥٩٦٩٦٣٩٤١٦٠٦	٧٥.٣٤٠٣٠٣٦٠٥٨٣٩٣٣٨	٧٥.٣٤٠٣٠٣٦٠٥٨٣٩٤
٣	٤٧.٧١١٣٩١٥٥٢١٠٣	٥٢.٢٨٨٦٠٨٤٤٧٨٩٦٥٥٧	٥٢.٦٤١٤٩٧١٠٨٩٧٤٦
	٤٢.٧٤١٤٩٣١٩٤٨٧٣	٥٧.٢٥٨٥٠٦٨٠٥١٢٧٣٣٢	٥٧.٢٥٨٥٠٦٨٠٥١٢٧٠
	٣٨.٢٨٩٢٨٨٥٩٧٥١١	٦١.٧١٠٧١١٤٠٢٤٨٨٧٩٨	٦١.٧١٠٧١١٤٠٢٤٨٨٨
	٣٤.٣٠٠٨٥١٧٤١٨٧١	٦٥.٦٩٩١٤٨٢٥٨١٢٩٣١٣	٦٥.٦٩٩١٤٨٢٥٨١٢٩٣
	٣٠.٧٢٧٨٧٣٨٦٠١١٣	٦٩.٢٧٢١٢٦١٣٩٨٨٦٨٧٩	٦٩.٢٧٢١٢٦١٣٩٨٨٦٩
	٢٧.٥٢٧٠٧٨٣٠٨٩٧٥	٧٢.٤٧٢٩٢١٦٩١٠٢٤٧٦٦	٧٢.٤٧٢٩٢١٦٩١٠٢٤٨
	٢٤.٦٥٩٦٩٦٣٩٤١٦١	٧٥.٣٤٠٣٠٣٦٠٥٨٣٩٣٣٨	٧٥.٣٤٠٣٠٣٦٠٥٨٣٩٠

Table ١ shows the results. When b = ٢, the corresponding asset price is about ٧٣٨.٩٠. Thus option values corresponding to S > ٧٤٠ are not available. As shown in Table ١, the proposed method can give very accurate solution of option values corresponding to the asset prices which are near a. It is clear that the number of accurate solution of the option values is largely affected by the choice of the right boundary $x = b$. To obtain ١٣ accurate solutions, "b" must be ١.٨. Thus with an appropriate choice of "b", the number of accurate solution will be more.

٦. Conclusion

We study an efficient numerical approximation of the American option pricing. The exact artificial boundary condition is derived for an unbounded domain by implementing in one practical example. Numerical results show that the number of very accurate solution will be more when the artificial boundary is chosen appropriately.

Appendix A

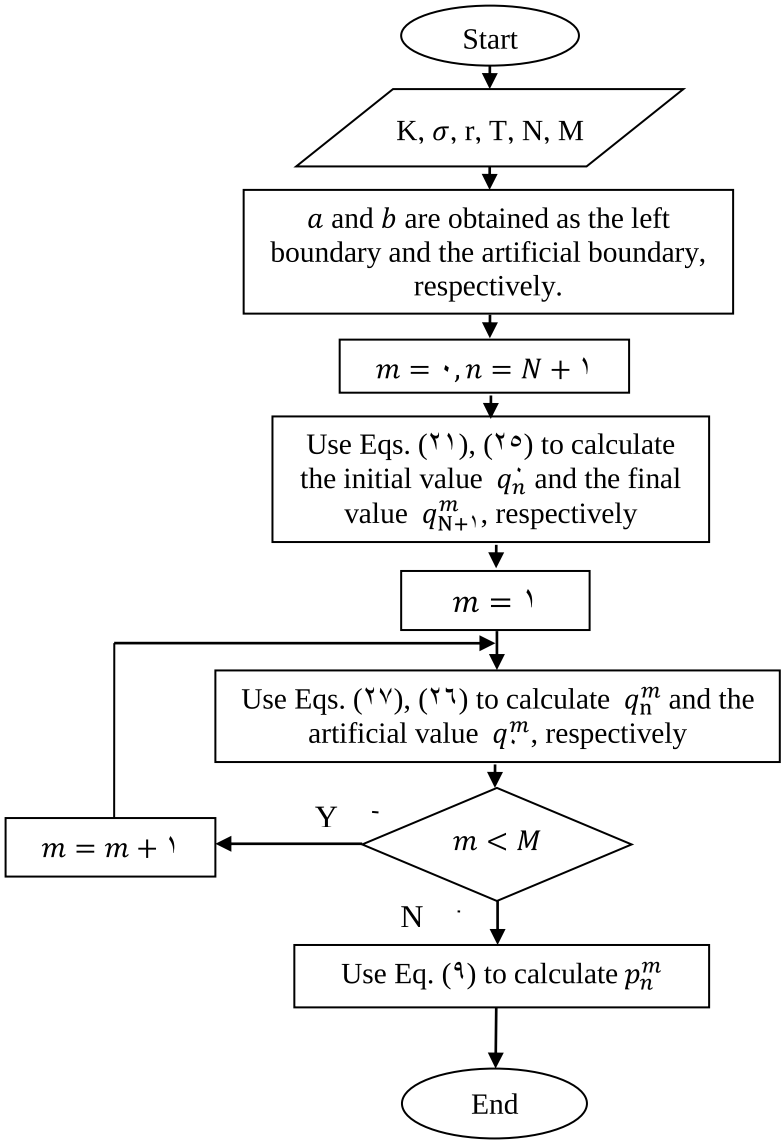

Fig. A.١. Schematic flow chart

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