Application of Stochastic Differential Games for Optimal Investment Strategy Selection

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Abstract:

In game theory, differential games are a group of problems related to the modeling and analysis of conflict in the context of a dynamical system. The problem usually consists of two actors, a pursuer and an evader, with conflicting goals. The dynamics of the pursuer and the evader are modeled by systems of differential equations. Differential games are related closely with optimal control problems. In an optimal control problem there is single control u(t) and a single criterion to be optimized; differential game theory generalizes this to two controls u(t), v(t) and two criteria, one for each player. Each player attempts to control the state of the system so as to achieve his goal; the system responds to the inputs of both players.
In this paper, a stochastic differential equation, approach to a risk-based, optimal investment problem of an insurer is discussed. A simplified continuous-time economy with two investment vehicles, namely, a fixed interest security and a share, is considered. The insurer's risk process is modeled by a diffusion approximation to a compound Poisson risk process. The goal of the insurer is to select an optimal portfolio so as to minimize the risk described by a convex risk measure of his/her terminal wealth. The optimal investment problem is then formulated as a zero-sum stochastic differential game between the insurer and the market.

Keywords: Optimal investment, Stochastic differential equation, Zero-sum stochastic differential game.

1. Introduction

Stochastic control has its wide applications in manufacturing, communication theory, signal processing, and wireless networks; see for example Kushner and Dupuis (2001), Fleming and Soner (2006) and references therein. On the other hand, zero-sum stochastic differential games, as the theory of two controller, extends the control theory into more realistic problems. Many problems arising in, for example, pursuit evasion games, queueing systems in heavy traffic, risk sensitive control, and constrained optimization problems, can be formulated as two-player stochastic differential games.

2. Static games

Let U and Z be two sets, and $P(u,z)$ a bounded function on $U \times Z$ called the game payoff. There are two controllers, one of which chooses $u \in U$ and wishes to minimize P. The other controller chooses $z \in Z$ and wishes to maximize P.
We consider two controller, zero sum differential games on a finite time interval $[t, t_1]$. The state of the differential game at time s is $\mathrm{x(s)} \in \mathbb{R}^{\mathrm{n}}$, which satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{ds}}\mathrm{x(s)} = \mathrm{G}\big(\mathrm{s}, \mathrm{x(s)}, \mathrm{u(s)}, \mathrm{z(s)}\big), \qquad \mathrm{t} \leq \mathrm{s} \leq \mathrm{t_1}$$

with initial data

$$x(t) = x_0$$

At time s, $u(s)$ is chosen by a minimizing controller and $z(s)$ is chosen by a maximizing controller, with $u(s) \in U$, $z(s) \in Z$. The sets U, Z are called the control spaces. The game payoff is

$$P(t,x;u,z) = \int_t^{t_1} L\big(s, x(s), u(s), z(s)\big)ds + \psi(x(t_1))$$

We make the following assumptions:

a. $U \subset \mathbb{R}^{m_1}$, $Z \subset \mathbb{R}^{m_2}$ and U , Z are compact;
b. G, L are bounded and continuous on $\overline{Q_0} \times U \times Z$;
c. There exist constants K_G, K_L such that
$|G(t,x,u,z) - G(t,y,u,z)| \leq K_G|x - y|$
$|L(t,x,u,z) - L(t,y,u,z)| \leq K_L|x - y|$
for all $t \in [t, t_1], x, y \in \mathbb{R}^{\mathrm{n}}, u \in U, z \in Z$
d. ψ is bounded and Lipschitz continuous.

Value function is

$$W(t,x;u,z) = \inf_{u \in U} \sup_{z \in \mathrm{Z}} P(t,x;u,z)$$

3. The model dynamics

We consider a simplified continuous-time economy with two investment assets, namely, a fixed interest security B and a share S. These securities are traded continuously over time in a finite time horizon $\mathcal{T} \coloneqq [0, T]$, for $T < \infty$. To model uncertainty, we consider a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where P represents a reference probability measure from which a family of, (real-world), probability measures is generated.
Let r be the constant continuously compounded risk-free rate of interest, where $r > 0$. Then the price process $\{B(t) | t \in \mathcal{T}\}$ of the bond B evolves over time as:

$$B(t) = e^{rt}, \qquad B(0) = 1. \tag{1}$$

Suppose μ and σ are the appreciation rate and the volatility of the share S, respectively. Let $\{W_1(t) | t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$ with respect to the P-augmentation of its own natural filtration, denoted by $F^{W_1} \coloneqq \{\mathcal{F}^{W_1}(t) | t \in \mathcal{T}\}$. Then the price process $\{S(t) | t \in \mathcal{T}\}$ of the share S is governed by a geometric Brownian motion, (GBM):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_1(t), \qquad S(0) = s_0 > 0. \tag{2}$$

We now model the risk process of an insurer. A diffusion risk process is considered here. This process is an approximation to the classical compound Poisson risk model. Let μ_Y and $\sigma_Y$be the drift and volatility of the diffusion risk process, where $\mu_Y \in \Re$ and $\sigma_Y > 0$. Suppose $W_2 := \{W_2(t)|t \in \mathcal{T}\}$ is a second standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$ with respect to the P-augmentation of its own natural filtration, denoted by $F^{W_2} := \{\mathcal{F}^{W_2}(t)|t \in \mathcal{T}\}$. We assume that

$$Cov\big(W_1(t), W_1(t)\big) = \rho_{12}t.$$

Then the diffusion risk process$R := \{R(t)|t \in \mathcal{T}\}$ is defined by:

$$R(t) = u_0 + \mu_Y t + \sigma_Y W_2(t)$$

Here u_0 is the initial surplus of the insurer. We assume that the insurer sets a premium rate according to the expected value premium principle. Let κ be the relative security loading, where $\kappa > 0$. Then the premium rate, (i.e. premium per unit of time), associated with the loading κ is determined as:

$$c(\kappa) = (1 + \kappa)\mu_Y.$$

Let $F^R := \{\mathcal{F}^R(t)|t \in \mathcal{T}\}$ and $F^S := \{\mathcal{F}^S(t)|t \in \mathcal{T}\}$ be the P-completed, right-continuous natural filtration generated by the insurance risk process R and the share price process S, respectively.
Define, for each $t \in \mathcal{T}$, the enlarged σ-field $\mathcal{g}(t) := \mathcal{F}^R(t) \vee \mathcal{F}^S(t)$, the minimal σ-field generated by both $\mathcal{F}^R(t)$ and $\mathcal{F}^S(t)$. For each $t \in \mathcal{T}$, the insurer has knowledge about the price process of the share S and the insurance risk process R up to time t. Consequently, the insurer has access to information described by $\mathcal{g}(t)$ to time t. Write $G := \{\mathcal{g}(t)|t \in \mathcal{T}\}$, so G describes the observable flow of information.
Let $\pi := \{\pi(t)|t \in \mathcal{T}\}$, where $\pi(t)$ represents the amount of money the insurer invests in the share at time t. Suppose $V^\pi := \{V^\pi(t)|t \in \mathcal{T}\}$ is the surplus process of the insurer with investments in the bond B and the share S. Then the evolution of the surplus process V^π is governed by:

$$dV^\pi(t) = \big(\kappa\mu_Y + rV^\pi(t) + \pi(t)(\mu - r)\big)dt + \sigma\pi(t)dW_1(t) - \sigma_Y dW_2(t) \tag{3}$$
$$V^\pi(0) = v_0 \tag{4}$$

To simplify the notation, we write $V(t) = V^\pi(t), t \in \mathcal{T}$, unless otherwise stated.
Write, for each $t \in \mathcal{T}$, $\sigma\big(\pi(t)\big) := (\sigma\pi(t), \sigma_Y)' \in \Re^2$and $\boldsymbol{W}(t) := \big(W_1(t), W_2(t)\big)' \in \Re^2$, where $\boldsymbol{y}'$ is the transpose of a matrix, or a vector, $\boldsymbol{y}$. Then the evolution of the surplus process of the insurer over time can be written as:

$$dV^\pi(t) = \big(\kappa\mu_Y + rV^\pi(t) + \pi(t)(\mu - r)\big)dt + \sigma'\big(\pi(t)\big)d\boldsymbol{W}(t)$$
$$V(0) = v_0.$$

A portfolio process π is said to be admissible if it satisfies the following conditions:

(1) π is G-progressively measurable;
(2) $\int_0^T [\pi(t)]^2 dt < \infty, \quad \mathcal{P} - a.s;$
(3) the surplus process V^π has a unique strong solution;
(4) $\int_0^T \big(|\kappa\mu_Y + rV^\pi(t) + \pi(t)(\mu - r)| + \|\sigma'\big(\pi(t)\big)\| + \|\sigma'\big(\pi(t)\big)\|^2\big)\, dt < \infty, \qquad \mathcal{P} - a.s.$

Write $\mathcal{A}$ for the space of admissible portfolio processes of the insurer.

4. Risk-based optimal investment problem

In this section we first state the risk-based optimal investment problem, where the object is to minimize the risk described by a convex risk measure of the terminal wealth of the insurer. We then formulate the risk-based optimal investment problem as a zero-sum, two-person, stochastic differential game between the insurer and the market.
Let $\mathcal{s}$ be the space of all lower-bounded random variables on the measurable space $(\Omega, \mathcal{g}(t))$. The space $\mathcal{s}$ consists of random variables describing financial positions whose values are realized at the terminal time T . Then a convex risk measure is defined as follows.

Definition. A convex risk measure ρ is a functional $\rho: \mathcal{s} \longrightarrow \Re^1$ such that it satisfies the following three properties:
(1) If $X \in \mathcal{s}$ and $K \in \Re$, then $\rho(X + K) = \rho(X) - K$.
(2) For any $X_1, X_2 \in \mathcal{s}$, if $X_1(\omega) \leq X_2(\omega)$, for all $\omega \in \Omega$, then $\rho(X_1) \geq \rho(X_2)$.
(3) For any $X_1, X_2 \in \mathcal{s}$, if $\lambda \in (0,1)$, $\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$.

The last property is the convexity, which reflects the situation that the risk of a trading portfolio might increase in a nonlinear fashion with the size of the portfolio because of the liquidity risk of a large portfolio. We impose the standard normalized assumption that $\rho(0) = 0$. With this assumption and the translation invariance property, (i.e. the first property), $\rho(X)$ can be interpreted as the minimum amount of capital which is required to make the financial position described by X acceptable.

As in Mataramvura and Øksendal (2007) [3], we consider a penalty function η of the following form:

$$\eta(Q_\theta, \pi) \coloneqq E\left[\int_0^T \lambda\big(t, \boldsymbol{Y}(.), \pi(t), \theta(t)\big)dt + h(\boldsymbol{Y}(T))\right], \quad (\pi, \theta) \in \mathcal{A} \times \Theta.$$

Where Θ is the space of admissible strategies θ and for each $\theta \in \Theta$, a new probability measure Q_θ absolutely continuous with respect to P on $\mathcal{g}(t)$.

It was shown in Delbean, Peng, and Rosazza Gianin (2008) [4], that if
(1) $h(\boldsymbol{y}) = 0$, for all $y \in \Re^2$,
(2) λ is independent of $\boldsymbol{Y}(.)$ and $\pi(t)$,
(3) λ is lower semi-continuous,

then the penalty function $\eta(Q_\theta, \pi)$ is the same as the representation of the penalty function of a dynamic convex risk measure, which satisfies the continuity and the time-consistent properties.

Then we specify a convex risk measure for the terminal wealth of the insurer as:

$$\rho\big(V^\pi(T)\big) \coloneqq \sup_{\theta \in \Theta}\left\{E^\theta[V^\pi(T)] - \eta(Q_\theta, \pi)\right\}$$

The goal of the insurer is to select an optimal portfolio process $\pi \in \mathcal{A}$ so as to minimize the risk described by $\rho\big(V^\pi(T)\big)$. That is, the optimization problem of the insurer is:

$$\Phi(v_0) \coloneqq \inf_{\pi \in \mathcal{A}} \rho\big(V^\pi(T)\big) \coloneqq \inf_{\pi \in \mathcal{A}}\left\{\sup_{\theta \in \Theta}\{E^\theta[V^\pi(T)] - \eta(Q_\theta, \pi)\}\right\}$$

Using a version of the Bayes' rule and the form of the penalty function,

$$\Phi(v_0) := \inf_{\pi \in \mathcal{A}} \sup_{\theta \in \Theta} E\left[-Y_1(T)Y_2(T) - \int_0^T \lambda\big(t, \boldsymbol{Y}(.), \pi(t), \theta(t)\big)dt - h(\boldsymbol{Y}(T))\right] = \Phi(\boldsymbol{y}), \ \ say.$$

Here $\boldsymbol{Y}(0) = \boldsymbol{y} = (y_1, y_2) \in \mathcal{D} \subset \Re^2$ and $\mathcal{D}$ is a bounded domain in $\Re^2$.
Write, for each$(\pi, \theta) \in \mathcal{A} \times \Theta$,

$$J^{\pi,\theta}(\boldsymbol{y}) := E^y\left[-Y_1(T)Y_2(T) - \int_0^T \lambda\big(t, \boldsymbol{Y}(.), \pi(t), \theta(t)\big)dt - h(\boldsymbol{Y}(T))\right].$$

Here E^y is the conditional expectation given $\boldsymbol{Y}(0) = \boldsymbol{y}$ under P. Then

$$\Phi(\boldsymbol{y}) = \inf_{\pi \in \mathcal{A}} \sup_{\theta \in \Theta} J^{\pi,\theta}(\boldsymbol{y}) \ = J^{\pi^*,\theta^*}(\boldsymbol{y})$$

This can be considered as a zero-sum, two-person, stochastic differential game between the insurer and the market. The market selects a probability measure indexed by θ, which corresponds to the worst-case scenario where the risk is maximal. The insurer then reacts by selecting an optimal portfolio strategy π so as to minimize the maximal risk. To solve the game problem, one must determine the optimal strategies π and θ of the insurer and the market, respectively, and the value function $\Phi(\boldsymbol{y})$.
The BSDE approach is used to solve the game problem. It leads to a simple and natural approach for the existence and uniqueness of an optimal strategy of the game problem. Unlike the HJB dynamic programming approach, the BSDE approach does not require Markov assumptions for controls and controlled processes.

5. Conclusion

The risk faced by the insurer was measured by a convex risk measure of the insurer's terminal wealth and the insurer selected an optimal portfolio so as to minimize the convex risk measure. The investment problem was formulated as a stochastic differential game. The BSDE approach is used to solve the game problem.

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