European Option Pricing with Transaction Costs

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Abstract
This paper deals with the construction of a finite difference scheme for a nonlinear Black-Scholes partial differential equation modelling stock option pricing in the realistic case when transaction costs arising in the hedging of portfolios are taken into account. The analysed model is the Barles-Soner one.

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1. Introduction
It is well known that Black-Scholes model is acceptable in financial markets where one assumes that volatility is observable or transaction costs are not taken into account. Under the transaction costs, the continuous trading required by the hedging portfolio is prohibitively expensive, [1]. Several alternatives lead to pricing models that are equal to Black-Scholes one but with an adjusted volatility denoted by σ,

\[ V_t + \frac{1}{2} \sigma(S,t,V_S,V_{SS})^2 S^2 V_{SS} + rS V_S - rV = 0, \quad s > 0, t \in [0,T], \quad (1.1) \]

where \( V \) is the option value that is a function of the underlying security \( S \) and the time \( t \). Here \( r \geq 0 \) denotes the riskless interest rate. There are some models for volatility \( \sigma \), [2, 3, 4]. A more complex model has been proposed by Barles and Soner [1], assuming that investor’s preferences are characterized by an exponential utility function. In their model the nonlinear volatility reads

\[ \sigma^2 = \sigma_0^2 (1 + \psi(t+\gamma a^2 S^2 V_{SS})), \quad (1.2) \]

where \( T \) is the maturity, and \( a = \mu \sqrt{\gamma N} \), with risk aversion factor \( \gamma \) and the number \( N \) of options to be sold. When \( a = 0 \), there is no transaction cost and classical Black-Scholes equation is recovered. The function \( \psi \) is the solution of the nonlinear initial value problem

\[ \psi'(A) = \frac{\psi(A) + 1}{2 \sqrt{A \psi(A) - A}} - 1, \quad A \neq 0, \quad \psi(0) = 0. \quad (1.3) \]
In this paper we deal with a European vanilla call option pricing equation (1.1) where \( \sigma \) is given by (1.2)-(1.3), together with final and boundary conditions taking the form

\[
V_t + \frac{1}{2}(\sigma(S,t,V_S,V_{SS}))^2S^2V_{SS} + rSV_S - rV = 0, \\
S > 0, t \in [0,T], \\
V(S,T) = \max(0, S - E), \quad S > 0, \\
V(0,t) = 0, \quad \lim_{s \to \infty} \frac{V(S,t)}{S - Ee^{-r(T-t)}} = 1.
\]  

(1.4)

Using the change of variable \( \tau = T - t \), \( U(S,\tau) = V(S,t) \) problem (1.4) except the asymptotic condition is transformed into

\[
U_\tau - \frac{S^2}{2}\sigma^2 U_{SS} - rSU_S + rU = 0, \quad 0 < S < \infty, 0 < \tau \leq T, \\
U(S,0) = \max(0, S - E).
\]

(1.5)

(1.6)

This paper is organized as follows. Section 2 includes some properties of the volatility correction function \( \psi \) after obtaining the solution of (1.3). In Section 3, by using semidiscretization with respect to \( S \) one gets a nonlinear system of ordinary differential equations with respect to the time, and then it is discretized using a forward explicit scheme. Section 4 includes conclusion.

2. Properties of the correction of volatility function

\( \psi(A) \) is an increasing function mapping the real line onto the interval \( ]-1, +\infty[ \) and is implicitly defined by

\[
A = \left( \frac{-\text{Arcsinh}\sqrt{\psi}}{\sqrt{\psi} + 1} + \sqrt{\psi} \right)^2, \quad \text{if} \quad \psi > 0, \quad (2.1)
\]

\[
A = -\left( \frac{\text{arcsin}\sqrt{-\psi}}{\sqrt{\psi} + 1} - \sqrt{-\psi} \right)^2, \quad \text{if} \quad -1 < \psi < 0. \quad (2.2)
\]

This has been shown in [5]. Also it can be seen in [1] that

\[
\lim_{A \to \infty} \frac{\psi(A)}{A} = 1, \quad \lim_{A \to -\infty} \psi(A) = -1. \quad (2.3)
\]

The next lemma will play an important role in studying the consistency of the numerical scheme. There is a proof of it in [6].

**Lemma.** Let \( \psi(A) \) be volatility correction function appearing in (1.3) verifying Eq. (1.5) and let \( g(A) = A\psi(A) \). Then \( \psi(A) \) is continuously differentiable at \( A = 0 \) and satisfies

\[
|g'(A)| \leq \max\{G, 2|A|\psi'(A_2) + d_2\}, \quad A \in \mathbb{R}
\]

(2.4)

where \( A_2 \) and \( d_2 \) are:

\[
A_2 = \left( \sinh 2 - \frac{1}{\sqrt{(\sinh 2)^2 + 1}} \right)^2 \simeq 9.58, \quad d_2 = \psi(A_2) - \psi'(A_2)A_2 \simeq 2.62 \quad (2.5)
\]

and \( A_1 \) and \( G \) are:

\[
A_1 = -\frac{(4\pi - 3\sqrt{3})^2}{36}; \quad G = \max\{|g'(A)|; A_1 \leq A \leq A_2\}. \quad (2.6)
\]
3. Semidiscretization and scheme construction

The computation of numerical solutions of the model is necessary because an exact solution is not available. The numerical analysis of the computed solution using finite difference methods for nonlinear models uses to be difficult and difficulties are overcome by means of linearization strategies that in some way falsify the model mainly near the maturity and the strike price. This fact motivates the search of an alternative numerical method which preserves the advantages of the finite difference method and that allows the full treatment of nonlinearities. The proposed method is semidiscretization method (SD). For partial we have:

\[
\frac{\partial U}{\partial S}(S_i, \tau) = \frac{U(S_{i+1}, \tau) - U(S_{i-1}, \tau)}{2h} + O(h^2), \quad (3.1)
\]

\[
\frac{\partial^2 U}{\partial S^2}(S_i, \tau) = \frac{U(S_{i-1}, \tau) - 2U(S_i, \tau) + U(S_{i+1}, \tau)}{h^2} + O(h^2), \quad (3.2)
\]

where \(S_j = E - L + jh, 1 \leq j \leq N - 1\) are the nodes of the underlying asset interval \([E - L, E + L]\), \(E\) is the strike price and \(L \leq E\) is the radius of the neighborhood about \(E\) where the numerical solution is computed. Let \(u(\tau) = [u_1, u_2, \ldots, u_{N-1}]^T\), be the approximation of theoretical values \(U(S_i, \tau)\), then we have:

\[
\frac{\partial u}{\partial \tau} = \frac{S_i^2 \sigma_i^2}{2h^2} u_{i-1} - \frac{2S_i \sigma_i^2}{h^2} u_i + \frac{S_i^2 \sigma_i^2}{2h^2} u_{i+1}.
\]

Define

\[
\alpha_i = \frac{S_i^2 \sigma_i^2}{2h^2} - \frac{rS_i}{2h},
\]

\[
\beta_i = -\frac{\sigma_i^2 S_i^2}{h^2} - r,
\]

\[
\gamma_i = \frac{\sigma_i^2 S_i^2}{2h^2} + \frac{rS_i}{2h}.
\]

So we have:

\[
\frac{\partial u}{\partial \tau} = \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1}.
\]

In particular:

\[
\frac{\partial u}{\partial \tau} |_{1} = \alpha_1 u_0 + \beta_1 u_1 + \gamma_1 u_2,
\]

\[
\frac{\partial u}{\partial \tau} |_{2} = \alpha_2 u_1 + \beta_2 u_2 + \gamma_2 u_3,
\]

\[
\vdots
\]

\[
\frac{\partial u}{\partial \tau} |_{N-1} = \alpha_{N-1} u_{N-2} + \beta_{N-1} u_{N-1} + \gamma_{N-1} u_N.
\]
Therefore
\[
\frac{du}{d\tau} = \begin{bmatrix}
\beta_1 & \gamma_1 & 0 & 0 & \ldots & 0 \\
\alpha_2 & \beta_2 & \gamma_2 & 0 & \ldots & 0 \\
0 & \alpha_3 & \beta_3 & \gamma_3 & \ldots & 0 \\
0 & 0 & \alpha & \gamma & \beta & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N-2 \\ u_N-1 \\ u_N \\ \end{bmatrix}
+ \begin{bmatrix}
\alpha_1 u_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{bmatrix}
\tag{3.4}
\]

By using second-order Lagrange interpolating polynomial:
\[
u_0 = 2u_1 - u_2, \quad u_N = 2u_{N-1} - u_{N-2}. \tag{3.5}
\]

(3.5) becomes
\[
\frac{du}{d\tau} = \begin{bmatrix}
\beta_1 + 2\alpha_1 & \gamma_1 - \alpha_1 & 0 & 0 & \ldots & 0 \\
\alpha_2 & \beta_2 & \gamma_2 & 0 & \ldots & 0 \\
0 & \alpha_3 & \beta_3 & \gamma_3 & \ldots & 0 \\
0 & 0 & \alpha & \gamma & \beta & \ldots & 0 \\
\alpha & \gamma & \beta & \ldots & \ldots & \ldots & 0 \\
\alpha & \gamma & \beta & \ldots & \ldots & \ldots & \ldots \\
\alpha & \gamma & \beta & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\begin{bmatrix}
u(\tau) 
\end{bmatrix}
\tag{3.7}
\]

We show (3.7) by:
\[
\frac{du}{d\tau} = M(\tau)u(\tau), \quad u(0) = (u_1(0), \ldots, u_{N-1}(0))^t, \quad u_i(0) = max(S_i - E, 0). \tag{3.8}
\]

Using Euler method
\[
u((n+1)k) = (I + M(nk))\nu(nk), \quad 0 \leq n \leq 1, \tag{3.9}
\]
the numerical solution of the vector initial value problem (3.8) one gets
\[
u(\tau) = \left( I^{n=0}_{n=1} (I + kM(nk)) \right) \nu(0), \tag{3.10}
\]
where \(k = \Delta \tau, \quad lk = \tau\). In [6] it can be seen that this scheme is consistent of order (2, 1) and conditionally time stable for appropriate fixed values of \(h = \Delta S\).

4. Conclusion

In this paper we consider a nonlinear Black-Scholes partial differential equation modeling stock option pricing in the realistic case when there are transaction costs. We use the Barles-Soner model to analyze. Semidiscretization method is proposed to preserve the advantages of the finite difference method and overcome the difficulties near the maturity and the strike price.
REFERENCES