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Numerical solution of Heun equation via linear stochastic differential equation

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Abstract
In this paper, the numerical approach of the following Stochastic differential equation which is named "Heun equation", will be represented.

$$\begin{cases} y'' + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)y' + \left(\frac{\mu}{x} + \frac{\nu}{x-1}\right)y = \xi, \\ y(0) = y_0, \;\; y'(0) = y_1. \end{cases}$$

such that $\alpha, \beta, \gamma, \mu, \nu$ and ξ, could be coefficients of Gaussian random numbers which is named wiener process. Making linear equations system from this equation, it could be solved by computing fundamental matrix of this system, with different methods. Finally, this stochastic equation is solved by numerical methods like E.M. and Milstein. Also its asymptotic stability and statistical concepts like expectation and variance of solutions are discussed.

Keywords and phrases: Heun Equation, Stochastic Differential Equation, Linear Equations System, Gaussian Random Numbers.

1. Introduction

Most of the known theoretical physics in nowadays is being involved in using a number of differential equations (O.D.E.or P.D.E.) in different orders [1, 2, 5, 6, 7]. If we intend to study just linear systems, various forms of the hypergeometric or the confluent and integrated hypergeometric equations often suffice to describe and identify this problem. In general case, these equations have power series solutions with simple relations between uninterrupted coefficients and can be generally represented in terms of simple integral transforms. In the case of nonlinear problem, we often utilize one form of the Painleve equation which is known as a linear second order differential equations.the classification of Differential equations is done according to their singularity structure around their singular points [11]. If a differential equation has no singularities all over the complex plane, it can merely be a constant. If the coefficient of the first derivative has at most single poles, and the coefficient of the term without a derivative has at most double poles when the coefficient of the second derivative is unity, this kind of second order differential equation has regular singularities. Then there exists a regular solution while extending vicinity these singular points. The second solution in neighbor of a regular singular point has a branch cut. If the poles of these coefficients are higher, we have irregular singularities and the general solution has essential singularities around these points [11]. For instance, the prominent differential equation

$$z\frac{d^2w}{dz^2} + (l+a)(\frac{dw}{dz}) = 0, \tag{1.1}$$

has two regular singular points, at $z = 0, \infty$. In physics, typical used equation is the following hypergeometric equation

$$z(1-z)\frac{d^2w}{dz^2} + (c - (l+a+b)z)\frac{dw}{dz} - abw = 0. \tag{1.2}$$

As we know, this equation has three regular singular points, at zero, one and infinity. Jacobi, Legendre, Gegenbauer and Tchebycheff equations are special cases of this second order equation. When the singular points at $z = 1$ and infinity are coalesced at infinity [11], we get the combined hypergeometric equation

$$z\frac{d^2w}{dz^2} + (c-z)\frac{dw}{dz})aw = 0, \tag{1.3}$$

with a singularity at infinity and a regular one at zero. Bessel, Laguerre and Hermite equations can be changed to this type. An important property of all these equations is that they allow infinite series solutions about one of their regular singular points and a recurrent relation can be found between two consecutive and uninterrupted coefficients of the series. This fact allows us having an opinion about the general properties of the solution, as asymptotic behavior at far away points, the radius of convergence of the series, etc.

In mathematics, the local Heun function H $l(a, q; \alpha, \beta, \gamma, d; z)$ (Karl L. W. Heun 1889) is the solution of Heun's differential equation as a recently made equation was introduced in 1889 by Karl M. W. L. Heun [12], that is holomorphic and one at the singular point z=0. The local Heun function is called a Heun function, denoted Hf, if it is also regular at $z = 1$, and is called a Heun polynomial, denoted Hp, if it is regular at all three finite singular points$z = 0, 1, a$.

Heun's equation is a second-order linear ordinary differential equation (O.D.E) of the form [8, 9, 13]:

$$\frac{d^2w}{dz^2}+[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}]\frac{dw}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}w=0 \tag{1.4}$$

the condition $\epsilon=\alpha+\beta-\gamma-\delta+1$ is necessary to ensure regularity of the point at ∞.

The complex number q is named the accessory parameter. Heun's equation has four regular singular points $z=0,1,a$ and $z=\infty$ with exponents $(0,1-\gamma),(0,1-\delta),(0,1-\epsilon), and\ (a,\beta)$. Every second-order linear O.D.E on the extended complex plane with at most four regular singular points, such as the Lam equation or the hypergeometric differential equation, can be transformed into this equation by a change of variable.

This equation is discussed in the book edited by Ronveaux. Also we can see any equation with four regular singular points can be transformed to the following equation:

$$\frac{d^2w}{dz^2}+(cz+dz-l+ez-f)\frac{dw}{dz}-abz-qz(z-l)(z-f)w=0 \tag{1.5}$$

There is a relation between the constants given as$a+b+1=c+d+e$. Then this equation such as before has regular singularities at zero, one, f and infinity. If we try to obtain a solution in terms of a power series, one can not get a recursion relation between two consecutive coefficients. Such a relation exists between at least three coefficients. A simple solution as an integral transform also can be found in [3, 5]. For more information regarding heun equation, their various solutions and it's applications to theoretical physics issues,you can refer to M. Hortacsu [10].

2. Making Stochastic Differential System

In general case, consider the second order liner S.D.E.

$$y''=(A(t)+\alpha(t)\xi_1)y'+(B(t)+\beta(t)\xi_2)y+(C(t)+\gamma(t)\xi_3)$$

such that $\xi_i(i=1,2,3)$ are white noise as Gaussian random variables ($\xi_i\sim N(0,t)$). This equation could be written as the following linear system:

$$y_1'=y_2,\ \ y_2'=(A(t)+\alpha(t)\xi_1)y_2+(B(t)+\beta(t)\xi_2)y_1+(C(t)+\gamma(t)\xi_3) \tag{2.1}$$

$$\begin{pmatrix}y_1\\y_2\end{pmatrix}'=\left(\begin{pmatrix}0&1\\B(t)&A(t)\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}+\begin{pmatrix}0\\C(t)\end{pmatrix}\right)$$
$$+\left(\begin{pmatrix}0&0\\\beta(t)y_1&\alpha(t)y_2\end{pmatrix}\begin{pmatrix}0&0\\\lambda\gamma(t)&(1-\lambda)\gamma(t)\end{pmatrix}\right)\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix} \tag{2.2}$$

In this form, we have put ξ_3 as a convex combination of ξ_1,ξ_2.

Thus, this equation matrix form is made as follows:

$$dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW, \tag{2.3}$$

such that $\xi_i = \frac{dW_i}{dt}$ $(i = 1, 2, 3)$, and W_i is wiener process.

Now, we want to address the constriction of this problem solutions. At first, we express the existence and uniqueness theorem of linear S.D.E.s like above.

Theorem 2.1. *Suppose that $D(t).y + C(t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ and $F(t).y + E(t) : \mathbb{R}^n \times [0,T] \to \mathcal{M}^{m\times n}$ are continuous and satisfy the following properties:*

$$\begin{aligned}&(1)\ \|D(t).(y(t) - \hat{y}(t))\| \leq L.\|y(t) - \hat{y}(t)\|, \\ &\quad\ \|F(t).(y(t) - \hat{y}(t)\| \leq L.\|y(t) - \hat{y}(t)\|.\ \textit{for all}\ 0 \leq t \leq T, y, \hat{y} \in \mathbb{R}^n \\ &(2)\ \|D(t).y + C(t)\| \leq L|1 + \|y\||, \\ &\quad\ \|F(t).y + E(t)\| \leq L|1 + \|y\||.\ \textit{for all}\ 0 \leq t \leq T, x \in \mathbb{R}^n,\end{aligned} \tag{2.4}$$

for some suitable $L \in \mathbb{R}$.

Let $y_0 \in \mathbb{R}^n$ is a random variable such that $E[y_0^2] < \infty$ so, there exist a unique solution $y \in L_n^2(0,T)$ of the following S.D.E:

$$\begin{cases} dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW \\ y(0) = y_0. \qquad (0 \leq t \leq T) \end{cases} \tag{2.5}$$

where $W(0)$, is a m-dimensional Brownian motion [7].

If $\underset{0\leq t\leq T}{Sup} \{\|C(t)\|, \|E(t)\|, \|D(t)\|, \|F(t)\|\} < \infty$
then $D(t)y + C(t)$ and $F(t)y + E(t)$ satisfy the hypotheses which have been posed in uniqueness and existence theorem for linear S.D.E. provided $E[y_0^2] < \infty$ [1].

In special case, if C, D, E and F have continuous elements in $[0,T]$, they get their finite maximum values in this interval.

On account of existence and uniqueness solution of linear S.D.E., for instance in narrow sense, the linear S.D.E. and its explicit solution is :

$$\begin{cases} dy = (D(t).y + C(t))dt + E(t).dW \\ y(0) = y_0 \end{cases} \tag{2.6}$$

$$y(t) = \Phi(t)\Big(y_0 + \int_0^t \Phi(s)^{-1}(C(s)ds + E(s)dW)\Big), \tag{2.7}$$

where $\Phi(0)$ is the fundamental matrix of the following O.D.E. system:

$$\frac{d\Phi}{dt} = D(t).\Phi_0,\ \ \Phi(c) = I.$$

In other words, we have:

$$\begin{pmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\Phi}_{21} & \dot{\Phi}_{21} \end{pmatrix} = \begin{pmatrix} \Phi_{21} & \Phi_{22} \\ B(t)\Phi_{11} + A(t)\Phi_{21} & B(t)\Phi_{21} + A(t)\Phi_{22} \end{pmatrix}$$

and $\Phi_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i = j \end{cases}$.

Consequently it turns out two send-order equations by different initial conditions:

$$\begin{cases} \ddot{\Phi}_{11} = B(t)\Phi_{11} + A(t)\dot{\Phi}_{11} \\ \Phi_{11}(0) = 1, \dot{\Phi}_{11}(0) = 0 \end{cases}, \quad \begin{cases} \ddot{\Phi}_{12} = B(t)\Phi_{12} + A(t)\dot{\Phi}_{12} \\ \Phi_{12}(0) = 0, \dot{\Phi}_{12}(0) = 1 \end{cases} \tag{2.8}$$

so the explicit solution infers as follows:

$$\begin{pmatrix} y_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \dot{\Phi}_{11} & \dot{\Phi}_{12} \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} + \int_0^t \frac{1}{\det\Phi} \begin{pmatrix} \dot{\Phi}_{12} & -\Phi_{12} \\ -\dot{\Phi}_{11} & \Phi_{11} \end{pmatrix} \begin{pmatrix} 0 \\ C(s) \end{pmatrix} ds + \begin{pmatrix} 0 & 0 \\ \lambda\gamma d(t) & (1-\lambda)\gamma(t) \end{pmatrix} \begin{pmatrix} dW_2 \\ dW_1 \end{pmatrix}.$$

Therefore we should have this equality for equation solution:

$$\begin{aligned} y_1 = y = \Phi_{11} y(0) + \Phi_{12} y'(0) + \Phi_{11} \int_0^t \frac{1}{\det\Phi} (-\Phi_{12})(C(s)ds \\ + \lambda\gamma(s)dW_2 + (1-\lambda)\gamma(s)dW_1) \\ + \Phi_{12} \int_0^t \frac{1}{\det\Phi} (\Phi_{11})(C(s)ds + \lambda\gamma(s)dW_2 + (1-\lambda)\gamma(s)dW_1). \end{aligned} \tag{2.9}$$

The equation (2.8) are second-order linear O.D.E. we could solve them by various methods like series solution respect to nonsingular point or Frobinios series with respect to regular points. Also, we could apply *Sinc* method to solve directly this equation or convert it to a linear system equation and solve it by 4th-order $Rung - kutta$ method.

Afterwards, we decide to compute from equality (2.9) that it could be done by numerical methods like E.M. predictor-corrector E.M. and Milstein.

Also in matrix form which is convenient for $Matlab$ software, we could get the following recursive procedure.

$$y(t) = \Phi(t)\Big(y_0 + \int_0^t \Phi(s)^{-1}(C(s)ds + E(s)dW_s)\Big)$$

$$\begin{cases} \Phi^{-1}(t_{i+1}).y(t_{i+1}) = y_0 + \int_0^{t_{i+1}} \Phi(s)^{-1}(C(s)ds + E(s)dW_s)\Big) \\ \Phi^{-1}(t_i)y(t_i) = y_0 + \int_0^{t_i} \Phi(s)^{-1}(C(s)ds + E(s)dW_s)\Big) \end{cases}$$

Consequently, we could have:

$$y(t_{i+1}) = y(i+1) = \Phi(t_{i+1})\Big(\Phi^{-1}(t_i)y_i + \int_{t_i}^{t_{i+1}} \Phi(s)^{-1}(C(s)ds + E(s)dW_s)\Big)$$

$$y(i+1) = \Phi(t_{i+1})\Phi(t_i)^{-1}(y_i + C(t_i)\Delta t_i + E(t_i)\Delta W_i) \qquad (2.10)$$

such that;

$$\delta W_i = W(t_{i+1}) - W(t_i) \cong \sqrt{\delta t_i}\xi_i \ \ (\xi_i \sim N(0,1))$$

The last approximation has been concluded from independent increment property of wiener process (for any $t, s \in [0,T]$; $W(t) - W(s) = W(t-s) \sim N(0, t-s)$)

of course, $W(t)$ could be computed by an infinite series of haar function with standard Gaussian have been written in based on this approximation. Although, it could be done according (12) too.

Finally, this issue should be said that for almost each W, the random trajectories of S.D.E.

$$\begin{cases} dy = (D(t).y + C(t))dt + (F(t).y + E(t))dW \\ y(0) = y_0 + \xi \end{cases}$$

Converge uniformly on interval $[0,T]$ as $\begin{cases} \xi \to 0, \\ \epsilon = F(t).y + E(t) \to 0 \end{cases}$ to the trajectory of deterministic O.D.E. $\begin{cases} \dot{y} = D(t).y + C(t) \\ y(0) = y_0. \end{cases}$

In general case, the following theorem indicates this asymptotic stability for Linear stochastic systems.

Theorem 2.2. *(Dependence on parameters)Suppose for $k = 1, 2, \cdots$ that $D^k(t)+C^k(t)$ and $F^k(t)y+E^k(t)$ satisfy the hypothesis of existence and uniqueness theorem, with the same constant L which said as a real bond in theorem. Assume further that*

$$\lim_{k\to\infty} E(\|y_0^k - y_0\|) = 0$$

and for each $M > 0$, such that $\|y\| \le M$,

$$\lim_{k\to\infty} \max_{0\le t\le T}\Big(\|D^k(t)-D(t)\|+\|C^k(t)-C(t)\|+\|F^k(t)-F(t)\|+\|E^k(t)-E(t)\|\Big) = 0$$

Finally suppose that $y^k(0)$ solves:

$$\begin{cases} dy^k = (D^k(t) + C^k(t))dt + (F^k(t)y + E^k(t))dW \\ y^k(0) = y_0^k \end{cases}$$

Then

$$\lim E\Big(\max_{0\le t\le T}\|y^k(t)-y(t)\|^2\Big)=0,$$

where y *is the unique solution of*

$$\begin{cases} dy=(D(t).y+C(t))dt+(F(t).y+E(t))dW,\\ y(0)=y_0.\end{cases}$$

Lemma 2.3. *According to asymptotic stability and solution of linear S.D.E. we could solve any nonhomogeneous linear O.D.E. by its correspond S.D.E.*

In addition, the analytic solution and least square error of O.D.E. could be found in bused on expectation and variance of S.D.E. solution.

Example 2.4. *Consider the following differential equation with noise*

$$\begin{cases}(1-x^2)y''-2xy'+n(n+1)y=f(x)\xi_1+g(x)\\ y(0)=y_0, y'(0)=y_1\end{cases}$$

the equivalent system equation is

$$dy=(D(t)y+C(t))dt+E(t)dW_t$$

such that:
$D(t)=\begin{pmatrix}0 & 1\\ \frac{-n(n+1)}{1-t^2} & \frac{2t}{1-t^2}\end{pmatrix}$, $C(t)=\binom{0}{g(t)}$, $E(t)=\binom{0}{f(t)}$, $y=\binom{y}{y'}$ *and* $\xi_t=\frac{dW}{dt}$.
According to linear S.D.E. solution, the answer is

$$y=\Phi(t)\Big(\begin{pmatrix}y_0\\ y_1\end{pmatrix}+\int_0^t\Phi^{-1}(t)(C(s)ds+E(s)dW_s)\Big), \tag{2.11}$$

$$\dot{\Phi}(t)=D(t).\Phi(t),\ \Phi(0)=I. \tag{2.12}$$

Consequently two second order equations get from this system equation.

$$\begin{cases}\Phi''_{1i}=\frac{2t}{1-t^2}\Phi'_{1i}-\frac{n(n+1)}{1-t^2}\Phi_{1i},\ (i=1,2)\\ \Phi_{1i}(0)=\delta_{1i},\ \Phi'_{1i}(0)=\delta_{2i}\end{cases} \tag{2.13}$$

We could solve these O.D.E.s by different methods like series solution, since functions. Also, this equation system could be solved by numerical methods of equation system like $Rung-kutta$ *from* $4th$ *order.*

Thus, recursive relation for (2.11) *is as follow:*

$$y_{i+1}=\Phi_{i+1}.\Phi_i^{-1}\big(y_i+C(t_i)\Delta t_i+E(t_i)\Delta W_i\big)$$

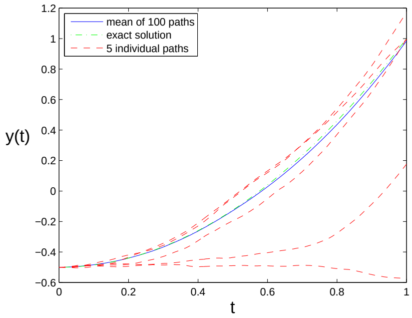

FIGURE 1. The exact solution with mean of $M = 100$ numerical solution by E.M. Method with $N = 2^9$ subinterval of $[0, 1]$ and also 5 individual numerical solution has been shown.In this case the least square error is:0.0234.

The equations (2.13) *are* $n-th$ *order legendre equations, for instance in particular case* $n = 2$*:*

$$\begin{cases} \Phi''_{11} = \frac{2t}{1-t^2}\Phi'_{11} - \frac{6}{1-t^2}\Phi_{11} \\ \Phi_{11}(0) = 1, \Phi'_{11}(0) = 0 \end{cases} \rightarrow \Phi_{11}(t) = -3t^2 + 1 \tag{2.14}$$

$$\begin{cases} \Phi''_{12} = \frac{2t}{1-t^2}\Phi'_{12} - \frac{6}{1-t^2}\Phi_{12} \\ \Phi_{12}(0) = 0, \Phi'_{12}(0) = 1 \end{cases} \rightarrow \Phi_{12}(t) = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + 0(t^7) \tag{2.15}$$

If we consider the following S.D.E:

$$\begin{cases} (1-t^2)y'' - 2ty' + 6y = 4t + \xi \\ y(0) = -\frac{1}{2},\ y'(0) = 0 \end{cases} .$$

it could be verified that $y = y_h + y_p = \frac{1}{2}(3t^2 - 1) + t$ *is the exact solution of corresponding O.D.E., without white noise. (it could be verified by Laplace transform method).*

On the other hand, S.D.E. solution is as follow:

$$y = \begin{pmatrix} y \\ y' \end{pmatrix} = \Phi.\left(\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \int_0^t \Phi^{-1}(s).\left[\begin{pmatrix} 0 \\ 4s \end{pmatrix} ds + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_s\right]\right)$$

For instance, if $g(t) = 0$*, we get* $y = -\frac{1}{2}(-3t^2 + 1) + (t - \frac{2}{3}t^3 + \frac{1}{5}t^5).W(t)$

Finally, $E[y] = \frac{-1}{2}(-3t^2 + 1)$ *and* $var(y) = t^2\Big(1 - \frac{2}{3}t^2 + \frac{1}{5}t^4.\Big)$

We have shown the maximum absolute errors in numerical solution of the example in the table for different values of N, where $\|LE^N_{EM}(h)\|$ are least squares errors for E.M. method and $\|LE^N_M(h)\|$ are least squares errors for

Milstein method.

N	$\|E_{SE}^{N}(h)\|$	$\|E_{DE}^{N}(h)\|$
2^7	5.3×10^{-2}	3.29×10^{-2}
2^8	2.40×10^{-2}	2.19×10^{-2}
2^9	2.34×10^{-2}	1.5×10^{-2}
2^{10}	2.15×10^{-2}	4.32×10^{-3}

References

[1] L. Arnold, Stochastic Differential Equations, Theory and Applications, JohnWiley, (1974).

[2] D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Review, 43, 525-546, (2001).

[3] N. V. Krylov, Introduction to the Theory of Diffusion Processes, American Math Society, (1995).

[4] J. Lamperti, A simple construction of certain diffusion processes, J. Math. Kyoto, 161-170,(1964).

[5] H. McKean, Stochastic Integrals, Academic Press, (1969).

[6] B. K. Oksendal, Stochastic Differential Equations: An Introduction with Applications, 4th Ed., Springer, (1995).

[7] L. C. Evans, An Introduction to Stochastic Differential Equations, Version 1.2, (2004).

[8] R. S. Maier, The 192 Solutions of the *Heun* Equation, Compute. Math. Appl, 50, 133-143, (2005).

[9] N. Gurappa, P. K. Panigrahi, On polynomial solutions of *Heun* equation. arXiv.math-ph/0410015v1, (2004).

[10] M. Horta, *Heun* functions and their uses in Physics, Istanbul, Turkey, (2011).

[11] S. Slavyanov, W. Lay, Special Functions, A Unified Theory Based on Singularities, Oxford University Press, Oxford, (2000).

[12] R. S. Borissov, P.P. Fiziev, Exact Solutions of Teukolsky Master Equation with Continuous Spectrum, Bulg. J. Phys, 37, 65-89, (2010).

[13] A. Ronveaux, *Heun*'s Differential Equations, Oxford University Press, Oxford, (1995).