Portfolio optimization problem with default risk

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Abstract

In this paper, we consider a stochastic portfolio optimization problem with default risk on an infinite time horizon. An investor dynamically chooses a consumption rate and allocates the wealth into the securities: a perpetual defaultable bond, a money market account with the constant return and a default-free risky asset. The goal is to choose the optimal investment to maximize the infinite horizon expected discounted power utility of the consumption policies (controls). The default risk premium and the default intensity are assumed to rely on a stochastic factor formulated by a diffusion process. We study the optimal allocation and consumption policies to maximize the infinite horizon expected discounted non-log HARA utility of the consumption, and we use the dynamic programming principle to derive the Hamilton–Jacobi–Bellman (HJB) equation. Then we explore the HJB equation by employing a so-called sub–super solution approach. The optimal allocation and consumption policies are obtained in terms of the classical solution to a PDE. Finally, we get an explicit formula for the optimal control strategy. In this article The solutions are then used in portfolio management subject to default risk and derive the optimal investment and consumption policies.

Keywords: Portfolio optimization, Default risk, HJB equation, consumption policies,

1. Introduction

Merton proposed the strategy that maximizing the total expected discounted utility of the consumption for a market investment problem. Fleming and Pang discussed a classical Merton portfolio optimization problem, where the interest rate $r$ was assumed to be an ergodic Markov diffusion process. Bielecki and Jang studied an optimal allocation problem associated with a defaultable risky asset and there the goal was to maximize the expected HARA utility of the terminal wealth. Hou and Jin employed an intensity-based approach for the defaultable market and assumed that each investor receives a proportion of the market value of the debt prior to the default if a default occurs. Jang suggested a dynamics for the price of a defaultable bond, and studied the expected discounted utility of the wealth when the default risk premium and intensity were assumed to be constants. In this article, we investigate a portfolio optimization problem with default risk, and suggested a dynamics for the price of a defaultable bond, and studied the expected discounted utility of the wealth when the default risk premium and intensity were assumed to be constants. In this article, we investigate a portfolio optimization problem with default risk. An investor dynamically chooses a consumption rate and allocates the wealth into the securities: a perpetual defaultable bond, a money market account with the constant return and a default-free risky asset. Here the goal is to maximize the infinite horizon expected discounted utility of the consumption. There, the post-default HJB equation admitted a constant solution and the pre-default HJB equation is a linear uniformly elliptic equation with variable coefficients. For the non-log utility case, we find that the HJB equation is nonlinear. Due to its
nonlinearity, we adopt the so-called sub–super solution argument to study the equation. Finally, we get an explicit formula for the optimal control strategy.

2-The price dynamics of the financial securities

we shall present a model with the specifications of a reduced-form framework for an intensity-based defaultable market and of the dynamics of the financial securities (defaultable bond, money market account and defaultfree risky asset).

Let \((\Omega, F, P)\) be a complete real-world probability space and \(\tau\) be a nontrivial random time on the space. For \(t \geq 0\), let \(t \geq 0\) us define a default indicator process \((z_t)_{t \geq 0}\) by \(z_t = 1_{(\tau \leq t)}\).

Suppose that \((\omega_t, \bar{\omega}_t)_{t \geq 0}\) is a 2-dimensional standard Brownian motion on \((\Omega, F, P)\), and \(F = (F_t)_{t \geq 0}\) is the natural filtration of \((\omega_t, \bar{\omega}_t)_{t \geq 0}\). Let \(D_t = \sigma(z_u; 0 \leq u \leq t)\) and \(G_t = F_t \lor \mathcal{D}_t\) with \(t \geq 0\). Consider the conditional survival probability,

\[
S_t = P(\tau > t|F_t), \quad S_0 = 1 \quad (2.2)
\]

Assume that for each \(t > 0, S_t > 0\) a.s. and \(\mathbb{E}[S_t] > 0\). This implies that there is always a chance that the firm defaults.

Let \((1/\eta_t)_{t \geq 0}\) denote the default risk premium satisfying \(1/\eta_t \geq 1\) for all \(t \geq 0\), and \(\rho \in (0, 1)\) denote the constant loss rate when a default occurs. We can suggest the price dynamics \((p_t)_{t \geq 0}\) for a perpetual defaultable bond that pays constant coupon \(\tilde{C}\) per unit time as follows:

\[
dp_t = \rho p_t dt + \rho \lambda_t p_t (1 - z_t)(1/\eta_t - 1) dt - (1 - z_t) \tilde{C} dt - \rho p_t dm_t \quad (2.3)
\]

where \((\lambda_t)_{t \geq 0}\) is an F-adapted default intensity process, and \((m_t)_{t \geq 0}\) is a càdlàg \((P, \mathcal{F})\)-martingale.

The Methods of portfolio

Consider an investor who can access to a money market account \((\zeta_t)_{t \geq 0}\) with the constant interest rate \(r > 0\) and a default-free risky asset \((\beta_t)_{t \geq 0}\) with the evolutions:

\[
\begin{align*}
d\zeta_t &= r\zeta_t dt, \quad \zeta_0 = 1, \\
d\beta_t &= \beta \beta_t dt + a\beta_t d\omega_t, \quad \beta_0 = \beta > 0.
\end{align*}
\]

we use \((\gamma_t)_{t \geq 0}\) to describe a stochastic economic factor which evolves according to the following stochastic differential equation (SDE):

\[
d\gamma_t = \mu(\gamma_t) dt + d\bar{\omega}_t, \quad \gamma_0 = y,
\]

where the drift coefficient \(\mu(\cdot)\) is assumed to satisfy

\[
\begin{align*}
(1) \quad \mu(\cdot) &\in C_1(R) \quad \text{and there exist positive constants } C_1, C_2 \quad \text{such that} \\
(2) \quad C_2 \leq \mu_y(y) &\leq -C_1 < 0,
\end{align*}
\]

for all \(y \in R\) where \(\mu_y := \frac{\partial \mu}{\partial y}\).

Remark1: Let \(x_t\) be the total wealth at time \(t\), and \(\xi_t\) and \(\ell_t\) denote the respective \(t\)–time proportions in the wealth \(x_t\) of \((p_t)_{t \geq 0}\) and \((\beta_t)_{t \geq 0}\). Then \(1 - \xi_t - \ell_t\) is the \(t\)-time proportion in the wealth \(x_t\) of \((\zeta_t)_{t \geq 0}\). Assume that \((\eta_t)_{t \geq 0}\) is the consumption rate at time \(t\) assume that the default risk premium and the default intensity depend on \(t\), the economic factor at time \(t\), i.e., there exist a nonnegative measurable \(\lambda(\cdot)\) and a measurable \((0, 1]-valued \eta(\cdot)\) such that
$$\lambda_t = \lambda(y_t), \quad \eta_t = \eta(y_t), \quad t \geq 0.$$ 

The two technical assumptions are made.

(2) There exists a constant $C > 0$ such that $\sup_{y \in \mathbb{R}} \lambda(y)$.

(3) The constant $\eta_m := \inf_{y \in \mathbb{R}} \eta(y)$ is strictly positive.

Now, by the self-financial investment policy, the dynamics of the wealth process is described as

$$dx_t = x_t[r - c_t + (b - r)\ell_t + \rho \kappa_t(1 - z_t)\lambda(y_t)(1/\eta(y_t) - 1)]dt + a x_t \ell_t \sigma d\omega_t - \rho \kappa x_t dm_t \quad (2.4)$$

$$x_0 = x > 0.$$ 

In addition, under mild conditions, it follows from Itô’s rule that

$$x_t = \exp\left\{ \int_0^t r - c_s + (b - r)\ell_s + \rho \kappa_s(1 - z_s)\lambda(y_s)(p/\eta(y_s))ds \right\} \times \exp\left\{ \int_0^t a\ell_s d\omega_s - \frac{1}{2} \int_0^t a^2 \ell_s^2 ds \right\} \prod_{s \in t} \{1 - \rho \kappa \Delta z_s\}$$

is a unique strong solution of (2.4).

**The optimal portfolio with non-log HARA utility**

We aim to seek an optimal allocation pair $(\kappa_t, \ell_t)_{t \geq 0}$ and an optimal consumption rate $(c_t)_{t \geq 0}$ to maximize the infinite horizon expected discounted non-log utility of the consumption. If $(\kappa_t, \ell_t, c_t)_{t \geq 0}$ is admissible, then, wealth process $(x_t)_{t \geq 0}$ is strictly positive.

Let $U(x)$ be a non-log hyperbolic absolute risk aversion (HARA) type utility function given by

$$U(x) = \frac{1}{y} x^y \quad 0 < y < 1, \quad x > 0$$

For an admissible control $(\kappa, \ell, c)$ and an initial triple $(x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}$, Where $\alpha > 0$ be the discount factor.

Our purpose is to maximize

$$J(x, y, z, \kappa, \ell, c) = \mathbb{E}_{x,y,z} \left[ \int_0^{\infty} e^{-\alpha t} U(c_t x_t) dt \right]$$

for all admissible $(\kappa, \ell, c)$, and so the value function is,

$$v(x, y, z) = \max_{(\kappa, \ell, c) \in \mathcal{A}(\varnothing)} J(x, y, z, \kappa, \ell, c) \quad (2.4)$$

for $(x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}$. Then we explore the HJB equation by employing a so-called sub–super solution approach.

**The HJB equation**

Define the pre-default and post-default value functions by

$$v(0)(x, y) = v(x, y, 0) \quad \text{(the pre-default case)},$$

and

$$v(1)(x, y) = v(x, y, 1) \quad \text{(the post-default case)}.$$
By employing the Bellman principle, we obtain the following HJB equations associated with $v_{(0)}(x, y)$ and $v_{(1)}(x, y)$. Then it will turn out to be the classical solution to the HJB equation associated with the value function $v$. There, the post-default HJB equation admitted a constant solution and the pre-default HJB equation is a linear uniformly elliptic equation with variable coefficients. For the non-log utility case, we find that the HJB equation is nonlinear. Due to its nonlinearity, we adopt the so-called sub–super solution argument to study the equation.

**Solutions to the HJB equation**

We prove the existence of a classical solution to the HJB equation associated with the value function $v$ by using a sub–super solution approach. Let us start at defining

$$
\tilde{u}(y) = \log \tilde{w}(y) \quad \text{and} \quad \tilde{u}(y) = \log \tilde{w}(y).
$$

As a consequence, $\tilde{u}$ and $\tilde{u}$ respectively satisfy

$$
\frac{1}{2} \tilde{u}_{yy} + \frac{1}{2} \tilde{u}_y^2 + \mu(y) \tilde{u}_y + \frac{\gamma(b - r)^2}{2a^2(1 - \gamma)} + ry - \alpha + (1 - \gamma)e^{\tilde{u}} = 0 \quad (2.5)
$$

$$
\frac{1}{2} \tilde{u}_{yy} + \frac{1}{2} \tilde{u}_y^2 + \mu(y) \tilde{u}_y + \frac{\gamma(b - r)^2}{2a^2(1 - \gamma)} + ry - \alpha + \gamma \lambda(y) \frac{1}{\eta(y)} + (1 - \gamma) e^{\tilde{u}} = 0 \quad (2.6)
$$

Our aim in the subsection is to seek a classical solution $\hat{\theta}$ for the HJB equation associated with the value function $v$ and verify that $\hat{\theta}$ equals the value function defined in (2.4). In order to obtain $\hat{\theta}$,

$$
\hat{\theta}(x, y, z) = \frac{1}{\gamma} x^\gamma [z\tilde{u}(y) + (1 - z)e^\tilde{u}]
$$

is the desired classical solution to the HJB equation associated with the value function $v$. We get an explicit formula for the optimal control strategy.

**Lemma:** Suppose that

$$
\alpha > \frac{\gamma(b - r)^2}{2a^2(1 - \gamma)} + ry := Q
$$

Then (2.5) possesses a constant solution:

$$
\tilde{u}(y) \equiv (y - 1)[\log(\alpha - Q) - \log(1 - \gamma)] := C_p \quad (2.7)
$$

**The verification theorem**

Define

$$
\hat{\theta}(x, y, z) = \frac{1}{\gamma} x^\gamma [e^{zC_p + (1 - z)\tilde{u}(y)}]
$$

That is a classical solution of the HJB equation associated with the value function $v$.

Let $(x_t, \xi_t, \xi_t^*)_{t \geq 0}$ be a control policy given by
\(\ell_t^* = \frac{b - r}{a^2(1 - \gamma)}, \quad t \geq 0\)  \hspace{1cm} (2.8)

\[ c_t^* = c^*(y_t) = \begin{cases} \frac{u(y_t)}{e^{\gamma - 1}}, & 0 \leq t < \tau \\ \frac{c_p}{e^{\gamma - 1}}, & t \geq \tau \end{cases} \hspace{1cm} (2.9)\]

\[ \kappa_t^* = \kappa^*(y_t) = \begin{cases} \frac{1}{\rho} \left[ 1 - \frac{e^{u(y_t)}}{(y_t)e^{\gamma - 1}} \right], & 0 \leq t < \tau \\ 0, & t \geq \tau \end{cases} \hspace{1cm} (2.10)\]

**Verification Theorem.** Suppose that the conditions (2), (3), \(y < \eta_m\) of are satisfied and that \(y \leq \frac{1}{2}\). Let \((\kappa_t^*, \ell_t^*, c_t^*)_{t \geq 0}\) be defined in (2.8)–(2.9), respectively. Define a function on \((0, \infty) \times \mathbb{R} \times \{0, 1\}\) by

\[ \hat{\vartheta}(x, y, z) = \frac{1}{y} x \gamma \left[ e^{y \gamma + (1 - z)u(y)} \right] = \frac{1}{y} x \gamma \left[ z e^{u(y)} + (1 - z)e^{u(y)} \right] \]

(i) For all admissible control policies \((\kappa_t, \ell_t, c_t)_{t \geq 0} \in \mathcal{A}(\mathbb{G})\), it holds that

\[ \hat{\vartheta}(x, y, z) \geq \mathbb{E}_{x, y, z} \int_0^\infty e^{-a t} U(c_t x_t) \, dt \]

with \((x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}\). (ii) \((\kappa_t^*, \ell_t^*, c_t^*)_{t \geq 0} \in \mathcal{A}(\mathbb{G})\). Moreover, the value function \(\vartheta\) satisfies

\[ \vartheta(x, y, z) := \mathbb{E}_{x, y, z} \int_0^\infty e^{-a t} U(c_t^* x_t^*) \, dt = \hat{\vartheta}(x, y, z) \]

with \((x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}\). Here \((x_t^*)_{t \geq 0}\) denotes the wealth process (2.4) with \((\kappa_t, \ell_t, c_t)_{t \geq 0}\) replaced by \((\kappa_t^*, \ell_t^*, c_t^*)_{t \geq 0}\).

We carry out a sensitivity analysis for the optimal control strategy and the value function \(\hat{\vartheta}\), Verification Theorem by employing the sub–super solution of (2.7).

We try to discuss the parameter sensitivity of the optimal control \((\kappa_t, c_t)\) for the pre-default case. We try to discuss the parameter sensitivity of the optimal control \((\kappa_t, c_t)\) for the pre-default case. Then \(\kappa_t\) admits a lower bound:

\[ \kappa_t^* \equiv \frac{1}{\rho} \left( 1 - \frac{1}{\eta^{\gamma - 1}} \right), \quad 0 \leq t < \tau \]

At first, we analyze the relationship between the default risk premium \(1/\eta\) and the lower bound \(\kappa_t\). Since a higher default risk premium leads to a high yield, we guess that there is a positive relationship between the default risk premium and \(\kappa_t\). We also note that the slope of the curves decreases as the default risk premium increases.

Second, we analyze the relationship between the loss rate \(\rho\) and the lower bound \(\kappa_t\). Since a higher loss rate induces a higher potential loss, the investors will reduce their investment proportion of the defaultable bond. Then, we investigate the relationship between the risk aversion parameter \(\gamma\) and the lower bound \(\kappa_t\). Since the utility function has a constant Pratt’s measure of relative risk aversion \(1 - \gamma\). This implies that the investors with less risk aversion parameter detest risk much more and thus will reduce their investment proportion of the defaultable bond.

Finally, we consider the optimal consumption rate \(c_t\) for the pre-default case.
\[ c_t^* = e^{\frac{y_t}{r-1}}, \quad 0 \leq t < \tau \]

3. Main Result

We studied a stochastic portfolio optimization problem with default risk on an infinite time horizon. An investor dynamically chooses a consumption rate and allocates the wealth into the securities: a perpetual defaultable bond, a money market account with the constant return and a default-free risky asset. The goal was to maximize the infinite horizon expected discounted power utility of the consumption. The default risk premium and the default intensity were assumed to rely on a stochastic factor formulated by a diffusion process. We explore the corresponding HJB equation by employing a so-called sub–super solution approach. The optimal allocation and consumption policies were obtained in terms of the classical solution to a PDE. The results provided in this paper could be used in portfolio management subject to default risk.

References: