Pricing American Options by the Finite Element Method

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Abstract
In this paper we investigate the performances of high-order of finite element methods for American option pricing. First of all, the partial differential problem that yields the price of American options, which is a free boundary problem, is transformed to a problem with a fixed boundary by adding a suitable penalty term. Then, by employing a quadratic finite element method, a nonlinear system of differential equations is obtained which is solved using an ad-hoc implicit-explicit Euler time-stepping. Numerical results will be presented to demonstrate the validity and the effectiveness of the method proposed.

Keywords and phrases: American Option, Penalty Method, Finite Element Method, Explicit and Implicit Algorithm, Newton Method

1. INTRODUCTION

A very popular approach to derivative pricing is the use of mathematical models based on partial differential equations. In particular, among the most commonly employed models, there is the famous Black-Scholes (BS) model. The valuation and hedging of BS equation resulting in American-style option is no doubt a challenging topic in both academic and the financial industry. This stems from the facts that most liquidly traded options are American style contracts, which allow option holders to exercise their rights before maturity, and that there is no analytical solution to these financial products under a realistic situation so far.

In this paper, we are attempting to obtain high order accuracy for the American option pricing. Following [1,2], the FEM is carried out to the nonlinear obtained B.S equation, and the variational integrals are evaluated by Gauss-Lobatto quadrature, and the initial solution is collocated at Gauss-Lobatto nodes. In addition, the computational stock and time mesh is chosen such that one of the finite element boundaries is positioned on the strike price and guarantee stability of the implicit algorithm[2]. This strategy has already proven crucial to improve the convergence rates of some lattice-based numerical methods used to solve the BS model.
2. MATERIALS AND METHODS

Usually in financial literature, as a mathematical model for the movement of the asset price under the risk-neutral measure is considered a standard geometric Brownian motion diffusion process with constant coefficients $r$ and $\sigma$:

$$dS = rS \, dt + \sigma S \, dW$$  \hspace{1cm} (2.1)

where $S$ is the underlying stock price, $r$ - interest rate, $\sigma$ - volatility, $dW$ increments of Gauss-Wiener process. In this paper we consider the case of American option.

If $\tau$ is the time to expiry $T$ of the contract, i.e. $\tau = T - t$, $0 \leq t \leq T$, the price $V(S,t)$ of the option satisfies the Black-Scholes partial differential equation

$$\frac{\partial V(S,t)}{\partial t} - rS \frac{\partial V(S,t)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rV(S,t) = 0, \; S \in (0, b(t))$$  \hspace{1cm} (2.2)

which $b(t)$ is the optimal exercise boundary which is unknown. The equation (2.2) is endowed with initial and boundary conditions:

$$V(S,T) = \max(S - K, 0)$$  \hspace{1cm} (2.3)

$$V(S,t) = S - Ke^{-rt} \text{ as } S \to \infty \text{ and } V(0,t) = 0$$  \hspace{1cm} (2.4)

and with the additional condition for $S$

$$\lim_{S \to b(t)} V(S,t) = b(t) - K$$  \hspace{1cm} (2.5)

To get rid of free boundary $b(t)$, we add a suitable penalty term, which the B.S equation is formulated as follows:

$$\frac{\partial V(S, \tau)}{\partial t} - rS \frac{\partial V(S, \tau)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S, \tau)}{\partial S^2} + rV(S, \tau) + \frac{\epsilon C}{V(S, \tau) - V^* + \epsilon} = 0, \; S \in (0, \infty)$$  \hspace{1cm} (2.6)

where $C$ is constant parameter satisfying $C \geq rE$ and $V^* = S - E$ is defined as payoff.

Now we will present the FEM directly by considering the BS equation. The proposed problem’s domain in section (2) discretized by finite elements.

Converting the $S \in (0, \infty)$ to $y \in (0, 1)$ and implementing FEM on relation (2.6) and using integration by parts the B.S equation is rewritten as follows:
\[
\int_I \left[ \frac{\partial u(y,t)}{\partial t} + \frac{dz_1(y)}{dy} - z_2(y) \right] \frac{\partial u(y,t)}{\partial y} + ru(y,t)w_j(y) - z_1(y) \frac{\partial u(y,t)}{\partial y} \frac{\partial w_j(y)}{\partial y} \\
+ \frac{\epsilon C}{V(S,t) - (S_{\text{max}}yE) + \epsilon} dy = 0
\] (2.7)

which \( j \) goes from 1 to \( N \) and \( I \) denotes the domain \([0,1]\), and

\[
z_1(y) = 1 - \frac{\sigma^2}{2} \left( \frac{\varphi(y)}{\varphi'(y)} \right)^2
\] (2.8)

\[
z_2(y) = \frac{r}{\varphi(y)} - \frac{1}{2} \sigma^2 \left( \frac{\varphi^2(y)\varphi''(y)}{(\varphi'(y))^3} \right)
\] (2.9)

By considering

\[u(y,t) = \sum_{j=1}^{N_1} w_j(y)u_j(t).\] (2.10)

By defining the transpose vector of \( U(t) = [u_1(t), u_2(t), ..., u_N(t)] \), we construct an implicit algorithm for the relation (2.3) as follows:

\[
M \frac{U(t^{j+1}) - U(t^j)}{dt} + \frac{1}{2} (rM + H - K)(U(t^{j+1}) + U(t^j)) \frac{\epsilon C}{U(t^{j+1}) - (S_{\text{max}}y - E) + \epsilon} = 0,
\] 
\[j = 1, 2, ..., N(2.11)\]

and rearrange according in time we reach a nonlinear closed form:

\[F[U(t^{j+1}), U(t^j)] = 0, \ j = 1, 2, ..., N_1.\] (2.12)

The coefficient matrix of integral form of relation (15) for the shape functions \( w_j(y) \) are written as follows:

\[
M_{i,j} = \int_I w_i(y)w_j(y)dy 
\] (2.13)

\[
H_{i,j} = \int_I \left( \frac{dz_1(y)}{dy} - z_2(y) \right)w_i(y) \frac{dw_j(y)}{dy} dy
\] (2.14)

\[
K_{i,j} = \int_I z_1(y) \frac{dw_i(y)}{dy} \frac{dw_j(y)}{dy} dy
\] (2.15)

\[i = 2, 3, ..., N_1 - 1, \ j = 1, 2, ..., N_1.\]

By inserting boundary conditions in first and last column matrix coefficient of \( U(t^{j+1}) \), we implement the nonlinear implicit system (2.11) by Newton method as follows:

\[AU_{m+1}(t^{j+1}) = -J^{-1}[F(U_m(t^{j+1}), U_m(t^j))]F[U_m(t^{j+1}), U_m(t^j)] + U_m(t^{j+1}),
\] 
\[m = 1, 2, ..., M, j = 1, 2, ..., N_1.\] (2.16)
where $M$ is a fixed number and $J^{-1}(F)$ denotes the inverse of Jacobian matrix.

3. **Main Result**

In this section we directly focus onto the case of piecewise quadratic and cubic FEM but also other approximation spaces could be considered. Note that the number of time values are always chosen so that the stability of proposed implicit algorithm is kept [2].

We input data and parameters for American call option as: $E = 92$, $S_0 = 70$, $r = .05$, $T = .25$ and $\sigma = .3$.

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<th>N</th>
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Table 1: convergence of the quadratic FEM, American Call Option.

4. **Conclusion**

We have investigated the convergence of the FEM in solving the Black-Scholes model for American option pricing. We have found that, if the FEM is carried out and the mesh is aligned with the options payoff, then the computed solutions exhibit nodal convergence, even though the initial data are non-smooth. We presented numerical simulations (tables) in which the values of $h$ are halved and the ratios of the errors at the nodes obtained. This high convergency rates can be obtained using relatively low-order polynomial approximation spaces.

**References**
