Spread Option Pricing with Finite Liquidity

A. Shidfar\textsuperscript{*} Kh. Paryab\textsuperscript{*} A. R. Yazdanian\textsuperscript{*}

\textsuperscript{*}Iran University of Science and Technology
E-mail: yazdanian@iust.ac.ir
shidfar@iust.ac.ir,paryab@iust.ac.ir

Abstract
We discuss the pricing and hedging of European spread options on correlated assets when, in contrast to the standard framework and consistent with a market with imperfect liquidity, the option trader’s trading in the stock market has a direct impact on one of the stocks price. We consider a full feedback model in which price impact is fully incorporated into the model. In order to find the numerical solution of the model, we will first apply Matched Asymptotic Expansions techniques on the price impact function to expand the full nonlinear equation to combination of linear equations. Then we use the Peaceman-Rachford scheme as an alternating direction implicit method to solve the linear equations numerically. Finally we provide a numerical analysis of the effect of the illiquidity in the underlying asset market on the replication of a European option, such that compared to the Black-Scholes case, a trader generally buys more stock to replicate a option.

Keywords and phrases: Spread option, Price impact, Illiquid markets, Nonlinear finance, Asymptotic analysis, Peaceman-Rachford scheme.

1. Introduction
Classical asset pricing theory assumes that traders act as price takers, that is, the theory assumes that investors trades have no impact on the prices paid or received. The relaxation of this assumption and its impact on realized returns in asset pricing models is called liquidity risk. Consistent with this discussion, most of the option pricing models assume that an option trader cannot affect the price in trading the underlying asset to replicate the option payoff, regardless of her trading size. The papers of Black and Scholes \cite{1}, and much of the work undertaken in mathematical finance has been aimed at relaxing of this underlying assumption. This is reasonable only in a perfectly liquid market.

In presence of such a price impact, the most important issues is how the impact price can affects the replication of an option. In order to respond to this question, there are two step. First, whether an option is still perfectly replicable or not. Second, how the presence of impact price change the replicating costs. This encouraged researchers to develop the Black-Scholes model to models that involve the price impact due to a large trader who is able to move the price by his/her actions. An excellent survey of these research can be found in \cite{2}.

Most of the research works discussed how the impact price affects the replication of an option, the our purpose of this paper is to investigate the effects of imperfect liquidity on the replication of an European Spread option by a typical option trader. The hedger is assumed
to be aware of the feedback effect and so would change the hedging strategy accordingly, i.e., full feedback model.

Several Spread options are also traded in the markets, however, in this paper we will focus our interest on Oil Markets and more specifically one of the most frequently quoted Spread options which are crack Spreads. A crack Spread represents the differential between the price of crude oil and petroleum products (gasoline or heating oil). The underlying indexes comprise futures prices of crude oil, heating oil and unleaded gasoline. Details of crack Spread options can be found in the New York Mercantile Exchange (NYMEX) Crack Spread Handbook [3]. In the Oil markets with finite liquidity, trading does affect the underlying assets price in trading the underlying assets to replicate the Spread option payoff. In our study, we are going to investigate the affects of impact price on Spread option pricing in Oil markets, when trading affects only the crude oil price and not petroleum products.

In fact, we want to model the price of a crack Spread option by assuming that the the crude oil in a market with finite liquidity and the petroleum products in a perfectly liquid market are correlated.

This paper is organized as follows: Section 2 discusses the general framework we use. We will first apply an asymptotic expansion in order to expand the full nonlinear equation to combination of linear equations. Then we propose a numerical method for solving the linear equations. For the numerical solution of the multi-dimensional parabolic equations, we shall study splitting schemes of the Alternating Direction Implicit (ADI) type. The ADI has been described in [4]. The particular finite difference scheme we use in this paper is Peaceman and Rachford scheme which introduced in [5]. We also discuss the stability and convergence of the scheme. In Section 3, we carry out several numerical experiments and provide a numerical analysis of the model for European calls. Section 4 contains the concluding remarks.

2. Materials and Methods

Our model of a financial market, based on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbf{P})$ that satisfies the usual conditions, consists of two assets. Their prices are modeled by a two-dimensional Itô-process $S(t) = (S_1(t), S_2(t))$. All the stochastic processes in this work are assumed to be $\{\mathcal{F}_t\}_{t \geq 0}$-adapted. Their dynamics are given by the following stochastic differential equations, in which $W(t) = (w_1(t), w_2(t))$ is defined a two-dimensional standard correlated Brownian motion with correlation $\rho$, and $\{\mathcal{F}_t\}_{t \in [0,T]}$ being its natural filtration.

\begin{equation}
\frac{dS_i(t)}{S_i(t)} = \mu_i(t, S_i(t))dt + \sigma_i(t, S_i(t))dw_i(t); \quad i = 1, 2,
\end{equation}

where $\mu_i(t, S_i(t))$ and $\sigma_i(t, S_i(t))$ are the expected return and the volatility stock $i$ respectively in the absence of any trading by the trader. It is possible to add a forcing term, $\lambda(t, S_1(t))$ to the process for the first stock, which is dependent on the stock price and time, i.e.,

\begin{align*}
dS_1(t) &= \mu_1(t, S_1(t))S_1(t)dt + \sigma_1(t, S_1(t))S_1(t)dw_1(t) + \lambda(t, S_1(t))d\Delta_1(t), \\
dS_2(t) &= \mu_2(t, S_2(t))S_2(t)dt + \sigma_2(t, S_2(t))S_2(t)dw_2(t),
\end{align*}

where $\lambda(t, S_1(t))$ is the price impact function of the trader for the first stock. The term $\lambda(t, S_1(t))d\Delta_1(t)$ represents the price impact of the investors trading. We note that the classical Black-Scholes model is a special case of this model where $\lambda(t, S_1(t)) = 0$.

Our aim is to price a Spread Option in illiquid market with price impact of the trader stocks,
and the following payoff at maturity $T$ (a call at this case):

$$h(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - k)^+ \quad (2.3)$$

where $k$ is the strike price. The Spread option pricing model under illiquid markets and payoff function (2.3), for the case where the interest rate and the reference volatility are constant, takes the following generalized Black-Scholes pricing PDE

$$
\frac{\partial V}{\partial t} + \frac{1}{2(1-\lambda)} \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} \frac{\partial^2 V}{\partial S_2^2} + \frac{1}{1-\lambda} \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_1 \sigma_2 S_1 S_2 \lambda \frac{\partial^2 V}{\partial S_1 \partial S_2} + 2 \rho \sigma_1 \sigma_2 S_1 S_2 \lambda \frac{\partial^2 V}{\partial S_1 \partial S_2} + \rho \sigma_1 \sigma_2 S_1 S_2 \lambda \frac{\partial^2 V}{\partial S_1 \partial S_2} + r(V(S_1, S_2) - V_0) = 0, \quad 0 < S_1, S_2 < \infty, 0 < t < T,
$$

$$V(T, S_1, S_2) = h(S_1(S_2), 0 < S_1, S_2 < \infty), \quad (2.4)$$

Note that, consistent with standard Black-Scholes arguments, the drift of the modified process does not appear in the option pricing PDE, and also the classical Black-Scholes model for Spread option is a special case of this model where $\lambda = 0$.

If we now let $\varepsilon$ such that $\varepsilon > 0$, and consider $\lambda(\tau) = \varepsilon \lambda(\tau)$, and using the following regular expansion

$$V(t, S_1, S_2) \sim V^0(t, S_1, S_2) + \varepsilon V^1(t, S_1, S_2) + \ldots \quad (2.5)$$

we get the following equations

$$
\frac{\partial V^0}{\partial t} + \frac{\sigma_1^2 S_1^2 \sigma_1^2 V^0}{2 \partial S_1^2} + \frac{\sigma_2^2 S_2^2 \sigma_2^2 V^0}{2 \partial S_2^2} + \sigma_1 \sigma_2 S_1 S_2 \rho \frac{\partial^2 V^0}{\partial S_1 \partial S_2} + r[S_1 \frac{\partial V^0}{\partial S_1} + S_2 \frac{\partial V^0}{\partial S_2}] - rV^0 = 0,
$$

$$V^0(T, S_1, S_2) = h(S_1(S_2), 0 < S_1, S_2 < \infty), \quad (2.6)$$

and

$$
\frac{\partial V^1}{\partial t} + \frac{\sigma_1^2 S_1^2 \sigma_1^2 V^1}{2 \partial S_1^2} + \frac{\sigma_2^2 S_2^2 \sigma_2^2 V^1}{2 \partial S_2^2} + \sigma_1 \sigma_2 S_1 S_2 \rho \frac{\partial^2 V^1}{\partial S_1 \partial S_2} + r[S_1 \frac{\partial V^1}{\partial S_1} + S_2 \frac{\partial V^1}{\partial S_2}] - rV^1 = G,
$$

$$V^1(T, S_1, S_2) = 0, \quad 0 < S_1, S_2 < \infty, \quad (2.7)$$

where

$$G = \lambda(-2 \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V^0}{\partial S_1 \partial S_2} + \frac{\partial^2 V^0}{\partial S_1^2} - \sigma_1^2 S_1^2 (\frac{\partial^2 V^0}{\partial S_1^2})^2 - \sigma_2^2 S_2^2 (\frac{\partial^2 V^0}{\partial S_2^2})^2) \quad (2.8)$$

For the sake of notation, we write the following operators:

$$L = \frac{\partial}{\partial t} + A_x + A_y + A_{xy} \quad (2.9)$$
where
\[ A_x = \frac{1}{2} \sigma_1^2 x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r \Theta \]
\[ A_y = \frac{1}{2} \sigma_2^2 y^2 \frac{\partial^2}{\partial y^2} + r y \frac{\partial}{\partial y} - r(1 - \Theta) \]
\[ A_{xy} = \sigma_1 \sigma_2 xy \frac{\partial^2}{\partial x \partial y}, \]
and \(0 \leq \Theta \leq 1\). While symmetry considerations might speak for an \(\Theta = \frac{1}{2}\), it is computationally simpler to use \(\Theta = 0\) or \(\Theta = 1\), i.e. to include the \(rV\) term fully in one of the two operators. Hence, we can write
\[
\begin{aligned}
LV^0(t, x, y) &= 0 \\
LV^1(t, x, y) &= G
\end{aligned}
\]
(2.11)
In order to define a numerical solution to the equations, we need to truncate the spatial domain to a bounded area: \(\{(x, y); 0 \leq x \leq x_{\text{max}}, 0 \leq y \leq y_{\text{max}}\}\). Let us introduce a grid of points in the time interval and in the truncated spatial domain:
\[
\begin{aligned}
t_l &= l \Delta t, \quad l = 0, 1, ... L, \quad \Delta t = \frac{T}{L} \\
x_m &= m \Delta x, \quad m = 0, 1, ... M, \quad \Delta x = \frac{x_{\text{max}}}{M} \\
y_n &= n \Delta y, \quad n = 0, 1, ... N, \quad \Delta y = \frac{y_{\text{max}}}{N}.
\end{aligned}
\]
(2.12)
For the simplicity of notation, we assume that \(x_{\text{max}} = y_{\text{max}}\) and \(\Delta x = \Delta y\). Functions \(V^0(t, x, y)\) and \(V^1(t, x, y)\) at a point of the grid will be denoted as \(V_{mn}^0 = V^0(t_l, x_m, y_n)\) and \(V_{mn}^1 = V^1(t_l, x_m, y_n)\). If we need to refer to the solution at a specific time point, we will use notation \(V_{mn}^0 = V^0(t_l, x_m, y_n)\) and \(V_{mn}^1 = V^1(t_l, x_m, y_n)\). Furthermore, let symbols \(A_{dx}, A_{dy}\) and \(A_{dxdy}\) denote second-order approximations to the operators \(A_x, A_y\) and \(A_{xy}\). Since the differential operator can be split as in (2.10) we can use Alternating Direction Implicit (ADI). The general idea is to split a time step in two and to take one operator or one space coordinate at a time. We implement the Peaceman-Rachford scheme of this method.
\[
\begin{aligned}
(I - \frac{\Delta t}{2} A_{dx}) V_{mn}^{0,l+1/2} &= (I + \frac{\Delta t}{2} A_{dy}) V_{mn}^{0,l+1} + \frac{\Delta t}{2} A_{dxdy} V_{mn}^{0,l+1/2} \\
(I - \frac{\Delta t}{2} A_{dy}) V_{mn}^{0,l} &= (I + \frac{\Delta t}{2} A_{dx}) V_{mn}^{0,l+1/2} + \frac{\Delta t}{2} A_{dxdy} V_{mn}^{0,l+1/2}.
\end{aligned}
\]
(2.13)
In the first step of this scheme, we calculate \(V_{mn}^{0,l+1/2}\) using \(V_{mn}^{0,l+1}\). This step is implicit in direction \(x\). In the second step, defined by equations (2.13), we use \(V_{mn}^{0,l+1/2}\) to calculate \(V_{mn}^{0,l}\). This step is implicit in the direction of \(y\). So on, someone can obtain, the Peaceman-Rachford scheme for \(V^1\) in (2.7) as:
\[
\begin{aligned}
(I - \frac{\Delta t}{2} A_{dx}) V_{mn}^{1,l+1/2} &= (I + \frac{\Delta t}{2} A_{dy}) V_{mn}^{1,l+1} + \frac{\Delta t}{2} A_{dxdy} V_{mn}^{1,l+1/2} - \frac{\Delta t}{2} G_{l+1} \\
(I - \frac{\Delta t}{2} A_{dy}) V_{mn}^{1,l} &= (I + \frac{\Delta t}{2} A_{dx}) V_{mn}^{1,l+1/2} + \frac{\Delta t}{2} A_{dxdy} V_{mn}^{0,l+1/2} - \frac{\Delta t}{2} G_l
\end{aligned}
\]
(2.14)
using the Von Neumann analysis, a sufficient condition for stability of the scheme is
\[
\frac{\Delta t}{\Delta x^2} \leq \frac{A}{\max(\sigma_1, \sigma_2) x_{\text{max}}^2}.
\]
(2.15)
Thus, the Peaceman-Rachford scheme is stable if the number of steps in the time interval, $L$, and in the spatial domain, $M = N$, satisfy inequality (2.15). This condition is a consequence of the cross-derivative term in the formula for the amplification factor. In the absence of this term, the scheme would be unconditionally stable. The remaining issue we need to address is the convergence of the numerical method to the true value of the problem. According to [4], this scheme is first-order accurate in time and space and due to stability, therefore the scheme is convergence.

3. Main Result

In this section, we check the properties of the proposed numerical scheme for the equations (2.6)-(2.7). We fix the values of the parameters of the marginal dynamical equations according to Table 1. We also assume the following price impact form

$$\lambda = \begin{cases} 
\varepsilon(1 - e^{-\beta(T-t)^{3/2}}), & S < S_1 < S, \\
0, & \text{otherwise},
\end{cases}$$

where $\varepsilon$ is a constant price impact coefficient, $T-t$ is time to expiry, $\beta$ is a decay coefficient, $S_1$ and $S$ represent respectively, the lower and upper limit of the stock price within which there is an impact price. We consider $S_1 = 60, S = 140, \varepsilon = 0.01$ and $\beta = 100$ for the subsequent numerical analysis. Choosing a different value for $\beta, S_1$ and $S$ will change the magnitude of the subsequent results, however, the main qualitative results remain valid.

<table>
<thead>
<tr>
<th>$S(t_0)$</th>
<th>$\sigma$</th>
<th>$S_{min}$</th>
<th>$S_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>100</td>
<td>0.15</td>
<td>0</td>
</tr>
<tr>
<td>Asset 2</td>
<td>100</td>
<td>0.10</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 1.** Model data together with $r = 0.05$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\rho = 0.1$</th>
<th>$\rho = 0.5$</th>
<th>$\rho = 0.7$</th>
<th>$\rho = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Excess price</td>
<td>7.1600</td>
<td>6.2972</td>
<td>5.7956</td>
</tr>
<tr>
<td>$k = -5$</td>
<td>Excess price</td>
<td>5.3275</td>
<td>4.3645</td>
<td>3.7731</td>
</tr>
<tr>
<td></td>
<td>Excess price</td>
<td>4.2936</td>
<td>3.3638</td>
<td>2.7085</td>
</tr>
<tr>
<td>$k = 0$</td>
<td>Excess price</td>
<td>3.4027</td>
<td>2.4860</td>
<td>1.8642</td>
</tr>
<tr>
<td></td>
<td>Excess price</td>
<td>2.3395</td>
<td>1.4909</td>
<td>1.0319</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>Excess price</td>
<td>1.1267</td>
<td>0.5435</td>
<td>0.2593</td>
</tr>
<tr>
<td></td>
<td>Excess price</td>
<td>0.1905</td>
<td>0.0426</td>
<td>0.0088</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>Excess price</td>
<td>0.11267</td>
<td>0.0426</td>
<td>0.0088</td>
</tr>
<tr>
<td></td>
<td>Excess price</td>
<td>0.0426</td>
<td>0.01245</td>
<td>0.0029</td>
</tr>
<tr>
<td>$k = 10$</td>
<td>Excess price</td>
<td>0.0426</td>
<td>0.01245</td>
<td>0.0029</td>
</tr>
<tr>
<td></td>
<td>Excess price</td>
<td>0.0426</td>
<td>0.01245</td>
<td>0.0029</td>
</tr>
<tr>
<td>$k = 20$</td>
<td>Excess price</td>
<td>0.0426</td>
<td>0.01245</td>
<td>0.0029</td>
</tr>
</tbody>
</table>

**Table 2.** The values of a 0.4 year European call Spread option based on different correlation, and strike price structure. Excess price shows the difference in call Spread option from Black-Scholes. $m = l = 100$, values of the parameters used for these runs are given in Table 1.
4. Conclusion

In this work, we have investigated how the imperfect liquidity in the underlying asset market affects the replication of a Spread option. We applied Matched Asymptotic Expansions techniques on the price impact function and the Peaceman-Rachford scheme as an alternating direction implicit method to solve the problem numerically. The numerical results showed that the trader compared to the Black-Scholes generally borrows more (for a call) or lends more (for a put) due to the adverse price impact.

References