Valuing discretely monitored barrier options

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Abstract
Options with the barrier feature are considered to be the simplest types of path dependent options. Barrier options distinctive feature is that the payoff depends not only on the final price of the underlying asset, but also on whether the asset price has breached (one-touch) some barrier level during the life of the option. In this paper we explore the problem for pricing discrete barrier options utilizing the Black-Scholes model for the random movement of the asset price. We postulate the problem as a path integral calculation by choosing approach that is similar to the quadrature method. Also, we perform a numerical algorithm for fast and accurate valuation of the multi-dimensional integral that represents the formula for the double barrier option price. In addition, we present an error estimation of our approximation and derive for discrete barrier options an identity similar to the famous put-call parity. Our results for pricing discretely monitored one and double barrier options are in agreement with those obtained by other numerical and analytical methods in finance and literature.

Keywords and phrases: Discrete barrier options, Black-Scholes model, Quadrature method, Multi-dimensional integral.
1 Introduction

Barrier options common path-dependent options traded in the financial markets. They have a payoff that is dependent on the realized asset path via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low during the options life. For example, an up-and-out call option pays off the usual max$(S - K, 0)$ at expiry unless at any time during the life of the option the underlying asset has traded at a value $H$ or higher. In this example, if the asset reaches this level (from below, obviously) then it is said to "knockout" and become worthless. A part from "out" options like this, there are also "in" option switch only receive a payoff if a certain level is reached, otherwise they expire worthless. Barrier options are popular for a number of reasons. The purchaser can use them to hedge very specific cash flows with similar properties. Usually, the purchaser has very precise views about the direction of the market. If he or she wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying asset will be limited prior to expiry, then he may choose to buy an up-and-out all. It will be cheaper than a similar vanilla call, since the up side is severely limited. If he is right and the barrier is not triggered he gets the payoff he wanted. The closer that the barrier is to the current asset price then the greater the likelihood of the option being knocked out, and thus the cheaper the contract.

The derivation of the pricing formula for barrier options was pioneered by Merton [1] in his seminal paper on option pricing. Peter Carr gives closed form formulas and replication strategies for barrier options [2]. Analytical formulas using the method of images in the case of one barrier applied continuously are presented in[3]. A list of pricing formulas for one-asset barrier options and multi-asset barrier options both under the geometric Brownian motion (GBM) framework can be found in the articles by Rich [4] and Wong and Kwok [5]. [6] have discussed the oscillatory behavior of the Crank-Nicolson method for pricing barrier options, and they applied the backward Euler method in order to avoid unwanted oscillations. [7] presents an efficient algorithm to price the barrier options in the presence of proportional transaction costs, using the optimal portfolio framework, the barrier options prices are computed numerically by use of a Markov chain approximation to the continuous-time singular stochastic optimal control problem, for the case of exponential utility. In [8] developed a method of lines approach to evaluate the prices as well as the delta and gamma of the option. The method is able to efficiently handle both continuously monitored and discretely monitored barrier options and can also handle barrier options with early exercise features. The proposed algorithm differs from the recursive numerical integration procedure of AitSahlia, [9] and the tridiagonal probability algorithm of Wai, [10].

2 Model for discrete double barrier options

Barrier options are options whose main characteristic is that the payoff is initiated when the underlying asset price reaches a predetermined level during a certain period of time. Barrier options belong to the class of path-dependent options, this mean that its not only the value of the underlying asset price at maturity that is important, but also the path the underlying price has taken up to maturity.

One example of barrier options with a discrete monitoring clause is the following option:

**Definition 1.2.** A discrete double barrier knock-out call option is an option with a continuous payoff condition equal to max$(S - K, 0)$ which expires worthless if before the maturity the asset price has fallen outside the barrier corridor $[L, U]$ at the prefixed monitoring dates: at these dates the option becomes zero if the asset falls out of the corridor. If one of the barriers is touched by the asset price at the prefixed dates then the option is canceled, i.e. it becomes zero, but the holder may be compensated by a rebate payment.
In this paper we study only European options, i.e. they may be exercised only at expiry. We take options with no rebates, i.e. the option is zero at the monitoring dates if one of the barriers is touched by the asset price. Let \( L \) and \( U \) be the lower and the upper barrier, respectively, \( K \) is the strike price and the set \( B = \{ t_i \mid t_i \in [0,T] \} \) consists of the times when barriers are applied. We assume that the barriers are applied discretely and the barrier times \( t_i \) are distributed uniformly in the set \( B \).

The discrete monitoring is due to the fact that one trading year is considered to consist of 250 working days and a week of 5 days. Thus, taking for one year \( T = 1 \), the application of barriers occurs with a time increment of 0.004 daily and 0.02 weekly.

**The model:** Let \( \{ S_t, t \geq 0 \} \) follow the stochastic differential equation
\[
\frac{dS}{S} = \nu dt + \sigma dZ
\]
where \( Z \) is a standard Wiener process, \( \nu \) and \( \sigma \geq 0 \) and \( S_0 \) is fixed. Under the risk-neutral Black-Scholes formulation of constant risk-free interest rate \( r \), and constant volatility of return \( \sigma \), the price process \( \{ S_t \} \) of the underlying security is defined by
\[
S_t = S_0 e^{B_t}
\]
where \( B_t \) is the Brownian motion with instantaneous drift \( \hat{\mu} = (r - (\sigma^2/2)) \) and standard deviation \( \sigma \), respectively.

Let us denote with \( A_i = \{ S_t \in (L, U) \}, i = 1, 2, \ldots, m \) all the barrier events where \( S_t \) is the random asset price movement at time \( t_i = i\Delta t, t_i \in B \).

Pricing the discrete double barrier knock-out call option by valuating the expected payoff of the option at expiry \( T \) discounted to the present value at time \( t_0 \) The value of the discrete double barrier knock-out option is given by:
\[
e^{-rT} E [\max(S_T - K, 0)1_{\{A_1\}}, 1_{\{A_2\}}, \ldots, 1_{\{A_m\}}].
\]
where \( 1_{A} \) is the standard indicator function, \( S_T \) is the asset price at expiration time \( T \), \( K \) is the strike price, the initial moment is at zero time.

The discrete counterpart process can be defined as
\[
\tilde{S}_n = S_0 e^{D_n}, \quad n = 0, 1, \ldots, m
\]
where \( D_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n, D_0 = 0 \), and the \( \alpha_i \) are independent normally distributed random variables, i.e. \( N(\hat{\mu}, \hat{\sigma}^2) \), such that \( \hat{\mu} = (r - (\sigma^2/2))\Delta t \) and \( \hat{\sigma} = \sigma \sqrt{\Delta t} \), with \( \Delta t = T/m \) for the expiration date \( T \) and \( m \) number of the prefixed monitoring instants when the two barriers \( L \) and \( U \) are applied.

Using the model (4), formula (3) has the following form:
\[
e^{-rT} E [\max(S_0 e^{D_m} - K, 0)1_{\{A_1\}}, 1_{\{A_2\}}, \ldots, 1_{\{A_m\}}].
\]

**Theorem 2.1.** The value of double barrier knock-out call option monitored \( m \)-times is given by the value of the following m-dimensional integral:
\[
V(S, t) = e^{-rT} \int_0^{\ln \hat{\mu}} \int_0^{\ln \hat{\mu}} \cdots \int_0^{\ln \hat{\mu}} \int_{m \Delta t}^{\ln \hat{\mu}} (Le^{x_1 + x_2 + \cdots + x_m} - K) \\
\times f(x_1, x_2, \ldots, x_m) dx_1 \cdots dx_2 dx_1
\]
where the density function \( f(x_1, x_2, \ldots, x_m) \) is defined by
\[
\left( \frac{1}{\sigma \sqrt{2\pi \Delta t}} \right)^m e^{-\frac{(x_m-e)^2 + (x_{m-1}-e)^2 + \cdots + (x_2-e)^2 + (x_1-e)^2}{2\Delta t \sigma^2}}
\]
(6)
where \( c = (r - \frac{\sigma^2}{2}) \sqrt{\Delta t} \) and \( cc = c - \ln \frac{L}{S_0} \).

**Proof.** Using the formula (6) in this form in order to develop our numerical algorithm for approximating this multi-dimensional integral. If \( cc = c \) in (7), \( f(x_1, x_2, \ldots, x_m) \) is a multivariate normal probability density function of the variables \( x_i \), but the integral limits in (6) would be from \( \ln \frac{L}{S_0} \) to \( \ln \frac{L}{S_0} \) case of a down-and-out call option each integral has an infinit upper limit, [10].

Having in mind all the indicators, informula (3) we have:

\[
A_i = \{ S_i \in (L, U) \} = \{ S_i \Delta t \in (L, U) \}
\]  
(8)

or equivalently, dividing by \( S(0) > 0 \) and then take logarithm:

\[
\{ \ln \frac{S_i \Delta t}{S_0} \in (\ln \frac{L}{S_0}, \ln \frac{U}{S_0}) \} = \{ D_i \in (\ln \frac{L}{S_0}, \ln \frac{U}{S_0}) \}
\]

where \( D_i \) is defined as the sum of random variables \( \alpha_i \) in (4). In addition

\[
\begin{align*}
A_i &= \left\{ \left( D_i - \ln \frac{L}{S_0} \right) \in (0, \ln \frac{U}{S_0}) \right\}.
\end{align*}
\]  
(9)

Setting \( \tilde{\alpha}_1 - \alpha_1 - \ln \frac{L}{S_0}, \ c = (r - \frac{\sigma^2}{2}) \sqrt{\Delta t}, \ cc = c - \ln \frac{L}{S_0} \), we have that

\[
S_t = S_i \Delta t = S_0 e^{\tilde{\alpha}_1 + \alpha_2 + \cdots + \alpha_m}
\]

and using the conditions in (9), i.e. \( \tilde{\alpha}_1 \in (0, \ln \frac{L}{S_0}) \), \((\tilde{\alpha}_1 + \alpha_2) \in (0, \ln \frac{L}{S_0})\), \(\ldots, (\tilde{\alpha}_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m) \in (0, \ln \frac{L}{S_0})\), the problem of estimating the expectation (5) is finally reduced to evaluation of formula (6) in Theorem 2.1.

Analogously, the value of discrete double barrier knock-out put option:

\[
V_{Put}(S, t) = e^{-r t} \int_0^{\ln \frac{L}{S_0}} \cdots \int_0^{\ln \frac{L}{S_0}} \int_0^{\ln \frac{L}{S_0}} (Le^{x_1 + x_2 + \cdots + x_m} - K) \times f(x_1, x_2, \ldots, x_m) dx_m \cdots dx_2 dx_1
\]  
(10)

where the density function \( f(x_1, x_2, \ldots, x_m) \) is defined in (7).

Unfortunately, for large values of \( m \), this \( m \)-dimensional integral could not be quickly estimated on a computer. Experimentally, when \( m = 1, 2, 3 \) the integral could be estimated fast. For \( m = 4 \) the computations take a long period of time (minutes) while for \( m \geq 5 \) hardly a real machine could manage to finish the estimations with in a reasonable time.

However, the number \( m \) is the barrier observation frequency and usually it is 25 or 125 in case the option is observed weekly or daily, respectively. ■

### 3 Algorithm for valuation

In this section we suggest a quick numerical algorithm for pricing formula (6) and thus to overcome the time-obstacle that is frequently met in computations.

The main idea of the numerical algorithm is to substitute the continuous normally distributed random variables \( \alpha_i \) in (4) with discrete ones that are ‘normally distributed’ and instead of \( m \)-dimensional integral to evaluate \( m \) number of finite sums. Thus, the computations are substantially quicker.

To illustrate our algorithm let us turn to the beginning of our problem (5). If \( \epsilon_i \) are independent normally distributed random variables with mean 0 and variance 1, \( i = 1, 2, \ldots, m \), we have the system:

\[\text{We formally use the term normally distributed for a discrete random variable.}\]
Using the tabulated function \( \sigma \), we have

\[
L < S_{\Delta t} = S_0 e^{(r - \frac{\sigma^2}{2}) \Delta t + \sigma \sqrt{\Delta t}} < U
\]

\[
L < S_{\Delta 2t} = S_0 e^{(r - \frac{\sigma^2}{2}) 2\Delta t + (1 + \sigma^2) \sigma \sqrt{2\Delta t}} < U
\]

\[
\vdots
\]

\[
L < S_{\Delta m\Delta t} = S_0 e^{(r - \frac{\sigma^2}{2}) m\Delta t + (1 + \sigma^2 + \cdots + \sigma_m) \sigma \sqrt{m\Delta t}} < U.
\]

We divide each row of the system by \( S_0 > 0 \) and then take logarithm. After setting \( c = (r - \frac{\sigma^2}{2}) \Delta t \), we obtain

\[
\ln \frac{L}{S_0} < c + \epsilon_1 \sigma \sqrt{\Delta t} < \ln \frac{U}{S_0}
\]

\[
2\epsilon_1 \sigma \sqrt{\Delta t} < \ln \frac{U}{S_0}
\]

\[
\vdots
\]

\[
\ln \frac{L}{S_0} < mc + (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m) \sigma \sqrt{\Delta t} < \ln \frac{U}{S_0}.
\]

Subtracting \([\ln \frac{L}{S_0} + \frac{1}{2n} \ln \frac{U}{T}]\) and setting \( cc = c - [\ln \frac{L}{S_0} + \frac{1}{2n} \ln \frac{U}{T}]\), where \( n \) is an integer number, we have

\[
-\frac{1}{2n} \ln \frac{U}{T} < cc + \epsilon_1 \sigma \sqrt{\Delta t} < (\ln \frac{U}{T})(1 - \frac{1}{2n})
\]

\[
-\frac{1}{2n} \ln \frac{U}{T} < cc + c + (\epsilon_1 + \epsilon_2) \sigma \sqrt{\Delta t} < (\ln \frac{U}{T})(1 - \frac{1}{2n})
\]

\[
\vdots
\]

\[
-\frac{1}{2n} \ln \frac{U}{T} < cc + (m - 1)c + (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m) \sigma \sqrt{\Delta t} < (\ln \frac{U}{T})(1 - \frac{1}{2n}).
\]

Let \( \lambda_i = cc + \epsilon_1 \sigma \sqrt{\Delta t}, i = 2, \ldots, m \) and then \( \epsilon_1 = \frac{\lambda_1 - cc}{\sigma \sqrt{\Delta t}}, \, \epsilon_i = \frac{\lambda_i - cc}{\sigma \sqrt{\Delta t}}, \, i = 2, \ldots, m \). Setting \( d = \frac{1}{n} \ln \frac{U}{T} > 0 \) we obtain:

\[
-\frac{1}{2n} \lambda_i < (n - \frac{1}{2})d
\]

\[
-\frac{1}{2n} \lambda_i < (n - \frac{1}{2})d
\]

\[
\vdots
\]

\[
-\frac{1}{2n} \lambda_i + \cdots + \lambda_{m} < (n - \frac{1}{2})d.
\]

We have \( E(\lambda_1) = cc + E(\epsilon_1) \sigma \sqrt{\Delta t} = cc \) and \( E(\lambda_i) = c, \, i = 2, \ldots, m \) because \( \epsilon_i \in N(0, 1), \, i = 1, 2, \ldots, m \).

Then for the variances \( D(\lambda_i) \) of \( \lambda_1, \lambda_2, \ldots, \lambda_i, i = 1, 2, \ldots, m \), it is true that \( D(\lambda_i) = D(\epsilon_i)(\sigma \sqrt{\Delta t})^2 = \sigma^2 \Delta t \). And the density of \( \lambda_i \) are:

\[
P_i(x) = \frac{1}{\sqrt{2\pi D(\lambda_i)}} \exp \left( \frac{(x-cc)^2}{2D(\lambda_i)} \right) = \begin{cases} \frac{1}{\sqrt{2\pi \Delta t}} \exp \left( \frac{(x-cc)^2}{2\Delta t \sigma^2} \right) & \text{for } i = 1, \\ \frac{1}{\sqrt{2\pi \Delta t}} \exp \left( \frac{(x-cc)^2}{2\Delta t \sigma^2} \right) & \text{for } i = 2, \ldots, m. \end{cases}
\]

Then we replace the continuous random variable \( \lambda_i \) with a discrete one. Taking in mind the indicator \( 1_{\lambda_i} \) in (5), we are interested only in the following probabilities:

\[
P_0 = P(-\frac{1}{2} < \lambda_1 < \frac{1}{2})
\]

\[
P_1 = P((1 - \frac{1}{2})d < \lambda_1 < (1 + \frac{1}{2})d)
\]

\[
\vdots
\]

\[
P_k = P((k - \frac{1}{2})d < \lambda_1 < (k + \frac{1}{2})d) = \frac{1}{\sqrt{2\pi \Delta t}} \int_{(k - \frac{1}{2})d}^{(k + \frac{1}{2})d} \exp \left( \frac{(x-cc)^2}{2\Delta t \sigma^2} \right) dx
\]

\[
\vdots
\]

\[
P_{n-1} = P((n - 1 - \frac{1}{2})d < \lambda_1 < (n - 1 + \frac{1}{2})d).
\]

Using the tabulated function \( erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \), we define
\[ \mathcal{P}_k = \frac{1}{2} \text{erf}\left( \frac{x - c}{\sigma \sqrt{2\Delta t}} \right) \begin{array}{c} (k + \frac{1}{2})d \\ (k - \frac{1}{2})d \end{array} = f(k + 1) - f(k), \]

\[ f(k) := \frac{1}{2} \text{erf}\left( \frac{k \cdot d}{\sigma \sqrt{2\Delta t}} - \frac{d - c}{\sigma \sqrt{2\Delta t}} \right), \quad k = 0, 1, \ldots, n \]

\[ \lambda_1 = \begin{pmatrix} \cdots & 0 & d & 2d & \cdots & (n-1)d & \cdots \\ \cdots & \mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \cdots & \mathcal{P}_{n-1} & \cdots \end{pmatrix}. \]

Analogously, \( \lambda_2 \in (-kd, kd] \), where \( k = -(n-1), \cdots, -1, 0, 1, \ldots, n-1 \)

\[ q_k = P((k - \frac{1}{2})d < \lambda_2 < (k + \frac{1}{2})d) = \frac{1}{\sigma \sqrt{2\pi \Delta t}} \int_{(k-\frac{1}{2})d}^{(k+\frac{1}{2})d} e^{-\frac{(x-c)^2}{2\sigma^2}} \, dx = g(k+1) - g(k) \]

where like \( f(k) \) we define and estimate \( g(k) \) by the function \( erf(x) \) as

\[ g(k) := \frac{1}{2} \text{erf}\left( \frac{k \cdot d}{\sigma \sqrt{2\Delta t}} - \frac{d + c}{\sigma \sqrt{2\Delta t}} \right), \quad k = -n, \ldots, 0, \ldots, n \]

\[ \lambda_2 = \begin{pmatrix} \cdots & -(n-1)d & \cdots & -d & 0 & d & 2d & \cdots & (n-1)d & \cdots \\ \cdots & q_{-(n-1)} & \cdots & q_{(-1)} & q_0 & q_1 & q_2 & \cdots & q_{(n-1)} & \cdots \end{pmatrix}. \]

Here, it is important that we are interested only in the values of \( \lambda_2 \) lying in the interval \([- (n-1)d, (n-1)d] \) having in mind the sum \( \lambda_1 + \lambda_2 \). Thus, we use the probability:

\[ P(\lambda_1 + \lambda_2 = 0, \lambda_1 \in \{0, d, \ldots, (n-1)d\}, \lambda_2 \in \{-(n-1)d, \ldots, (n-1)d\}) \]

having in mind \( 1_{(\lambda_1)} \) and \( 1_{(\lambda_2)} \), we estimate the probabilities:

\[ \begin{align*}
\mathcal{P}(\lambda_1 + \lambda_2 = 0) &= b_0 = \mathcal{P}_0 q_0 + \cdots + \mathcal{P}_{n-1} q_{-(n-1)} = \sum_{j=0}^{n-1} \mathcal{P}_j q_{-j} \\
&\vdots \\
\mathcal{P}(\lambda_1 + \lambda_2 = kd) &= b_k = \mathcal{P}_0 q_k + \cdots + \mathcal{P}_{n-1} q_{-(n-1)} = \sum_{j=0}^{n-1} \mathcal{P}_j q_{-j} \\
&\vdots \\
\mathcal{P}(\lambda_1 + \lambda_2 = (n-1)d) &= b_{n-1} = \mathcal{P}_0 q_{n-1} + \cdots + \mathcal{P}_{n-1} q_0 = \sum_{j=0}^{n-1} \mathcal{P}_j q_{n-1-j}. 
\end{align*} \]

Suppose \( b_i = \mathcal{P}_i, (i = 0, 1, \ldots, n-1) \), we define the following random variable \( \lambda_1 + \lambda_2 \):

\[ \lambda_1 + \lambda_2 := \begin{pmatrix} \cdots & 0 & d & 2d & \cdots & (n-1)d & \cdots \\ \cdots & \mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \cdots & \mathcal{P}_{n-1} & \cdots \end{pmatrix}. \]

Applying the upper algorithm we find the random variable \( (\lambda_1 + \lambda_2) + \lambda_3 \), and analogously, the random variable \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m \), with

\[ \lambda := \begin{pmatrix} \cdots & 0 & d & 2d & \cdots & (n-1)d & \cdots \\ \cdots & \mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \cdots & \mathcal{P}_{n-1} & \cdots \end{pmatrix} \]

We have that: \( S_T = S_{0\Delta t} = S_0 e^{m \epsilon (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m) \sigma \sqrt{\Delta t}} \), and
\[ \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m = -\left[ \ln \frac{L}{S_0} + \frac{1}{2\sigma} \ln \frac{U}{L} \right] + mc + (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_m) \sigma \sqrt{\Delta t}. \]

Then

\[ S_T = S_0 e^{\lambda \left[ \ln \frac{L}{S_0} + \frac{1}{2\sigma} \ln \frac{U}{L} \right]} = Le^{\lambda \left[ \ln \frac{L}{S_0} + \frac{1}{2\sigma} \ln \frac{U}{L} \right]} = e^{(j + \frac{1}{2})d} \]

Using that \( d = \frac{1}{n} \ln \frac{U}{L} \). Thus, finally, we obtain the following value for (5):

\[ e^{-rT} \sum_{j=0}^{n-1} \mathcal{P}_j \max \left( Le^{(j + \frac{1}{2})d} - K, 0 \right) = e^{-rT} \sum_{j=j_0}^{n-1} \mathcal{P}_j \left( Le^{(j + \frac{1}{2})d} - K \right) \]

where \( j_0 = \left[ \frac{1}{2} \ln \frac{K}{L} + \frac{1}{2} \right] \) is evaluated by the function \([x]\).

Finally, the value of discrete double barrier knock-out call option is:

\[ \hat{V}(S, t) = e^{-rT} \sum_{j=j_0}^{n-1} \mathcal{P}_j \left( Le^{(j + \frac{1}{2})d} - K \right). \] (11)

Analogously, the value of discrete double barrier knock-out put option is:

\[ e^{-rT} \sum_{j=0}^{n-1} \mathcal{P}_j \max \left( K - Le^{(j + \frac{1}{2})d}, 0 \right) = e^{-rT} \sum_{j=j_0}^{n-1} \mathcal{P}_j \left( K - Le^{(j + \frac{1}{2})d} \right). \] (12)

**Estimation of the error.** If we denote the price of the discrete double barrier knock-out call option with \( V(S, t) \) given by formula (6) and with \( \hat{V}(S, t) \) the value (11) obtained by using the proposed numerical algorithm applied for \( n \) discrete points the error could be estimated as:

\[ V(S, t) - \hat{V}(S, t) = O \left( \frac{1}{n} \right) \] (13)

and thus a desired level of accuracy is very fast achieved.\(^2\) See [11].

Let denote with \( VV(S, t) \) the following expression

\[ VV(S, t) = e^{-rT} \sum_{i,j=0}^{n-1} \int \int_{G_{i,j}} \max(Le^{x+y} - K, 0)e^{-\frac{(x-a)^2+y^2}{2\sigma^2}} \, dx \, dy. \] (14)

and

\[ G_{i,j} = \begin{cases} 
  id < x < (i+1)d, & i = 0, 1, \ldots, n-1 \\
  (j-i-\frac{1}{2})d < y < (i+j+\frac{1}{2})d, & j = 0, 1, \ldots, n-1 
\end{cases} \]

Using relation (16) in [11], we prove that

\[ |V(S, T) - VV(S, T)| \leq e^{-rt} \frac{U - K}{2\pi D} \frac{U}{n} \ln \frac{U}{L} = O \left( \frac{1}{n} \right). \] (15)

\(^2\)The error of the Monte Carlo simulation is \( O \left( \frac{1}{\sqrt{M}} \right) \), where \( M \) is the number of simulations. Such a low rate of convergence is not quite desirable.[3].
Then to prove relation (13), it is sufficient to estimate:

\[ VV(S, T) - \tilde{V}(S, T) = \frac{e^{-rt}}{2\pi D} \sum_{i,j=0}^{n-1} \int \int_{G_{i,j}} \left( \frac{e^{-(x-a)^2 + (y-a)^2}}{2\pi} \right) \left( \max\{Le^{x+y} - K, 0\} \right) - \max\{Le^{(j+\frac{1}{2})d} - K, 0\} dx dy. \]

we obtain

\[ |VV(S, T) - \tilde{V}(S, T)| \leq C_n = O \left( \frac{1}{n} \right) \quad (16) \]

where \( C \) is some positive constant.

Finally, from (15) and (16) it follows relation (13) be cause we have that

\[ |V(S, T) - \tilde{V}(S, T)| \leq |V(S, T) - VV(S, T)| + |VV(S, T) - \tilde{V}(S, T)| \leq \frac{C}{n}. \]

Second, we admit that relation (13) is fulfilled for \( m = k \).

Third, the relation (13) for \( m = k + 1 \) is proved similarly the case \( m = 2 \).

4 Numerical results

In this section we present numerical results. We apply our algorithm to the most explored examples in literature for discrete barrier options that are discretely monitored daily and weekly. We have arranged the results in such order that the distance of the two barriers is increased with each subsequent example. This allows us to observe the numerical error that depends on the number of the discretization points between the barriers.

We have compared the results with those obtained by other numerical methods in Finance such as the Monte Carlo simulations [12], trinomial trees and the Crank-Nicolson method.

**Table 1** Prices of discrete double knock-out call option in 5 monitoring dates. The current price of the underlying asset is \( S_0 \), the strike price is 100, the volatility is 5% per annum, the call option has six months remaining to maturity, the risk-free rate is 0% per annum (compounded continuously), the lower barrier is placed at 95, and the upper barrier is imposed at 110, i.e. respectively \( S_0, K = 100, \sigma = 0.25, T = 0.5, r = 0.05, L = 95, U = 110 \).

<table>
<thead>
<tr>
<th>S(0)</th>
<th>Crank-Nicolson (N=1000)</th>
<th>Algorithm (N=200)</th>
<th>Monte Carlo (st. error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>0.1656</td>
<td>0.174503</td>
<td>0.174498</td>
</tr>
<tr>
<td>95.0001</td>
<td>\approx 0.1656</td>
<td>0.174501</td>
<td>0.174499</td>
</tr>
<tr>
<td>95.5</td>
<td>0.18429</td>
<td>0.182428</td>
<td>0.182901 (0.00056)</td>
</tr>
<tr>
<td>99.5</td>
<td>0.2218</td>
<td>0.229356</td>
<td>0.222934</td>
</tr>
<tr>
<td>100</td>
<td>0.2212</td>
<td>0.235214</td>
<td>0.232508</td>
</tr>
<tr>
<td>100.5</td>
<td>0.234978</td>
<td>0.234973</td>
<td>0.23410 (0.00073)</td>
</tr>
<tr>
<td>109.999</td>
<td>0.1658</td>
<td>0.167463</td>
<td>0.167462</td>
</tr>
<tr>
<td>110</td>
<td>0.1591</td>
<td>0.167398</td>
<td>0.167393</td>
</tr>
<tr>
<td>CPU</td>
<td>1 s</td>
<td>39 s</td>
<td>Hundred sec.</td>
</tr>
</tbody>
</table>

**Table 2** Prices of discrete double knock-out call option monitored daily (125 times) and weekly (25 times) for different \( v \)-values of the underlying asset \( S_0 \) and parameters \( S_0, K = 100, \sigma = 0.25, T = 0.5, r = 0.05, L = 95, U = 110 \).

<table>
<thead>
<tr>
<th>Price</th>
<th>Algorithm (N=200)</th>
<th>Monte Carlo (at, error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(0)</td>
<td>Daily</td>
<td>Weekly</td>
</tr>
<tr>
<td></td>
<td>125 ( 10^8 )-asset path</td>
<td>weekly 25</td>
</tr>
<tr>
<td>95</td>
<td>0.0027003</td>
<td>0.019528</td>
</tr>
<tr>
<td>95.0001</td>
<td>0.0027005</td>
<td>0.002673 (0.00007)</td>
</tr>
<tr>
<td>100</td>
<td>0.011394</td>
<td>0.011394 (0.00015)</td>
</tr>
<tr>
<td>109.999</td>
<td>0.00258415</td>
<td>0.002664 (0.44447)</td>
</tr>
<tr>
<td>110</td>
<td>0.0025843</td>
<td>-</td>
</tr>
<tr>
<td>CPU</td>
<td>39 s</td>
<td>Hundred sec.</td>
</tr>
</tbody>
</table>
5 Conclusions

The advantage of the presented algorithm is that it has a simple computer implementation and turns out to be very efficient in accuracy and speed for valuation of discrete double barrier knock-out options. This makes it a more competitive method than frequently described methods in Finance such as the quadrature method, the Monte Carlo simulations, and the Crank-Nicolson scheme. One more advantage of the algorithm is that it permits observing the entire life of the option that is a characteristic feature of the finite difference schemes. The presented algorithm works successfully both for one and double barrier knock-out options that are monitored discretely.

References